Birationally rigid singular Fano varieties

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by

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Abstract

Basic concepts of birational geometry are introduced. Then the techniques of the method of maximal singularities, a powerful tool of proving birational rigidity, is presented in detail. On this basis, it is proved that a Fano double space V of dimension m, branched over a generic hypersurface of degree $2m$, which is singular at a linear subspace of codimension k, is birationally superrigid if $2m \leq k(k-3)+4$. In particular, there are no non-trivial structures of a rationally connected fibre space on V, V is non-rational, and the groups of birational and biregular self-maps coincide.

Contents

0 Introduction

During the nineteenth century, algebraic geometry developed from many sources; firstly from the classical geometry itself, including projective geometry, and also the work carried out by the Italian school on curves and surfaces. Another motivation for the construction of this field of mathematics was in order to further develop the study of invariant theory, which played a key role in the development of abstract algebra (most notably by Hilbert and Emmy Noether) at the beginning of the twentieth century. The rigorous foundations for algebraic geometry were laid during these early decades of the twentieth century by van der Waerden, Zariski, and Weil. Classically, algebraic geometry is the study of systems of polynomial equations in several variables, with particular focus on understanding the geometric structure of solutions, that is, the set of zeros of polynomials, forming the objects of study known as algebraic varieties. Grothendieck's concept of scheme (1960's) extended the classical ideas to arbitrary commutative rings, with prime ideals the algebraic objects and schemes the geometric. Algebraic geometry is at the core of modern mathematics, having connections with areas as varied as complex analysis and number theory.

Birational geometry is the branch of algebraic geometry where the invariants are the function fields of the varieties; that is to say, two varieties are birational if and only if they have isomorphic function fields. More formally, a birational mapping from one algebraic variety to another is a rational mapping that has a rational inverse. Whereas in biregular geometry varieties are mapped by regular, that is, locally polynomial maps, here we have rational maps taking their place. In general, such rational maps are defined only at generic points, that is, on dense open subsets, failing to be defined on closed subsets where the map is said to be non-regular. In chapter 1, we show that the set of points at which a rational map is non-regular is of codimension ≥ 2. Blowups are the simplest birational maps, and are used to resolve the points of indeterminacy of a (bi)rational map (chapter 2).

In the context of birational geometry, the rationality problem is that of determining whether a given variety is birational to the projective space of the same dimension. For curves, rationality is determined by the genus: a curve is rational if and only if its genus is zero. For surfaces, differential geometric invariants can again be used to determine rationality (Castelnuovo rationality criterion). In dimension three and higher, the method of maximal singularities (chapter 4) is the main tool used to study the birational geometry of varieties, and first appeared in the paper

[9] - see section 3.3 for a discussion on how this developed from the earlier work of M. Noether and Gino Fano.

In example 5.4, we state some existing results concerning the birational superrigidity of double spaces; in particular, that a smooth double space of index 1 branched over a smooth hypersurface of degree $2m$ in the projective space $\mathbb{P}^m$ is birationally superrigid, and when the branch divisor contains isolated singularities (satisfying the given conditions), the same result holds. The main aim of this thesis is to extend these results by proving the birational superrigidity of a double space of index 1 branched over a generic hypersurface containing a singular locus of higher dimension.

The main geometric implications of birational superrigidity will then be that there are no non-trivial structures of a rationally connected fibre space on the variety, from which it immediately follows that the variety is non-rational, and that the groups of birational and biregular self-maps coincide (see corollary 5.3). It is this last property that distinguishes between a variety being rigid and superrigid. In theorem 4.24, we will see that for a birationally rigid primitive Fano variety, the group of birational self-maps of the variety is generated by the subgroup of biregular automorphisms together with the untwisting maps that we discuss in section 4.5. Hence, superrigidity is rigidity combined with property that the aforementioned groups coincide. The precise definitions of birational rigidity and superrigidity are given in chapter 4. In section 3.2 we explain why varieties with no non-trivial structures of a rationally connected fibre space are of interest.

The structure of this thesis is as follows. In chapter 1, we give a few preliminary results that we will need later on. We show that it follows from the fact that the local ring of a nonsingular point is a UFD that differential forms are birational invariants, and that there are no non-zero global differential forms on projective space. This property can then be used to immediately rule out rationality of varieties containing such forms (chapter 3). Chapter 2 describes the blowup of a variety, and the inverse image of a divisor (cycle) with respect to this blowup. In chapter 3, we consider properties of rationally connected (Fano) varieties and fibre spaces, and give a brief history of the rationality problem.

In chapters 4-6 the main result of this work is discussed and proved: the birational superrigidity of Fano double spaces $V \to \mathbb{P}^m$, $m \geq 6$, branched over a generic hypersurface W_{2m} of degree $2m$ with a large singular locus $P = \operatorname{Sing} W_{2m} \subset \mathbb{P}^m$, a linear subspace of high dimension. Our work is organised as follows: chapter 4 outlines how the proof of the main theorem will proceed, that is, by the method of maximal singularities, and the steps this entails are detailed in this section. Chapters 5 and 6 are based on a note written by the author [1]. In the former chapter, we formulate the problem, and prove the result for a very specific case for which we can use methods similar to those used to prove the result in the smooth case. We state what the main obstruction to proving the result in the general case is, and in chapter 6, we use the connectedness principle (inversion of adjunction) to complete the proof.

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The ground field is always an algebraically closed field of characteristic zero; the main example is $k = \mathbb{C}$.

1 Preliminaries

This chapter is divided into three main sections. In the first, we give the definition of a unique factorisation domain (UFD), and the goal of this section is to prove that the local ring $\mathcal{O}_x$ of a nonsingular point $x \in X$ is a UFD. We also state two important corollaries of this fact. Differential forms are introduced in the next section, and the UFD property of a local ring is used to show that differential forms make natural birational invariants. We also prove that there are no global differential forms on the projective space. In the final section, we define a linear system of effective divisors on a variety, and show how we can associate a rational map to a given system. Linear systems will be fundamental in proving the main result of this thesis.

1.1 Unique factorisation domains

Let X be a nonsingular variety, that is, for the maximal ideal $m_x \subset \mathcal{O}_x$, $\dim_k m_x/m_x^2 = \dim \mathcal{O}_x$ for each $x \in X$ so that the local ring $\mathcal{O}_x$ is a regular local ring for each $x \in X$. We begin by stating some standard algebraic definitions and results.

Definition 1.1 An integral domain R is a ***unique factorisation domain*** (UFD for short) if every non-zero non-unit element of R is expressible uniquely as a product of irreducible elements of R, up to permutation and multiplication by units. Furthermore, in a UFD, all irreducible elements are prime.

Definition 1.2 If k is a field, the ***ring*** $k[[x_1, \dots, x_n]]$ ***of formal power series*** in $x_1, \dots, x_n$ is the ring whose elements are infinite expressions of the form $\phi = F_0 + F_1 + F_2 + \dots$, where $F_i \in k[x_1, \dots, x_n]$ is a form of degree i.

In the ring of formal power series $k[[x_1, \dots, x_n]]$, we introduce a lexicographic ordering on monomials. That is, $x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} < x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$, if and only if $i_\alpha < j_\alpha$ for α the greatest number such that $i_\alpha \neq j_\alpha$. Hence,

$$1 < x_1 < x_1^2 < \dots < x_2 < x_1 x_2 < x_1^2 x_2 < \dots.$$

Theorem 1.3 (Weierstrass division theorem) *Let $s, q \in k[[x_1, \ldots, x_n]]$ with $q \neq 0$. Then there exist unique $p, r \in k[[x_1, \ldots, x_n]]$ such that*

$$s = pq + r,$$

where no monomial in r is divisible by the lowest monomial in q.

Proof This is the same as the polynomial division algorithm, but now its a transfinite algorithm. At each step of the algorithm, we subtract a monomial multiple of q from s, and hence we cancel the lowest monomial in r, which is divisible by the lowest monomial in q. □

Theorem 1.4 (Weierstrass preparation theorem) *If $q \in k[[x_1, \ldots, x_n]]$ and x_1^α is the lowest monomial in q for some integer α, then there exists an invertible element $u \in k[[x_1, \ldots, x_n]]$ and elements $a_0, \ldots, a_{\alpha-1} \in k[[x_2, \ldots, x_n]]$ such that*

$$uq = a_0 + a_1 x_1 + \ldots + a_{\alpha-1} x_1^{\alpha-1} + x_1^\alpha,$$

where the right hand side of this equality is called a Weierstrass polynomial.

Proof Substitute the given formal power series q and $s = x_1^\alpha$ into the division theorem to get

$$x_1^\alpha = uq + r, \tag{1}$$

where no monomial in r is divisible by x_1^α. Hence, we can write

$$a_0 + a_1 x_1 + \ldots + a_{\alpha-1} x_1^{\alpha-1} = -r, \tag{2}$$

where $a_i \in k[[x_2, \ldots, x_n]]$. The result is then given by adding (1) to (2). Note that the smallest monomial in u is 1, since $s = x_1^\alpha$ and x_1^α is the smallest monomial in q - this is the first step in the division algorithm. Hence, setting $q = u$ and $s = 1$ in the division algorithm, and using the definition of r, we obtain that $r = 0$, proving that u is indeed invertible. See [2, page 8] also. □

Theorem 1.5 *If R is an integral domain with unique factorisation, then so is the polynomial ring, $R[x]$.*

Theorem 1.6 *Suppose k is an infinite field. Then the formal power series ring $k[[x_1, \ldots, x_n]]$ is a UFD.*

Proof The proof is by induction on the number of variables $x_1, \ldots, x_n$, where the Weierstrass preparation theorem is used to reduce the statement of the theorem to the analogous statement for polynomials in x_1 with coefficients in $k[[x_2, \ldots, x_n]]$

That is, for a polynomial ring over a power series ring in fewer variables, which is a UFD by applying theorem 1.5 to the induction hypothesis. See, for example, [3, vol.2, chapter VII, §1, theorem 6]. □

Theorem 1.7 **(Krull's intersection theorem)** *For R a Noetherian local ring with maximal ideal M, we have* $\bigcap_i M^i = 0$.

Proof This is a corollary of Nakayama's lemma. See, for example, [4, appendix 6, proposition 4]. □

Theorem 1.8 **(D. Mumford)** *Let A be any Noetherian subalgebra of a formal power series algebra $\hat{A} = \mathbb{C}[[x_1, \ldots, x_n]]$ such that all the elements of A with constant term non-zero are invertible, and such that A contains the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. Then A is a UFD.*

We are interested in the case when $A = \mathcal{O}_x$, that is, the local ring of $\mathbb{C}[x_1, \ldots, x_n]$ at $(x_1, ..., x_n)$. We write $\hat{\mathcal{O}}_x$ for the ring of formal power series, and view $\mathcal{O}_x$ as a subring $\mathcal{O}_x \subset \hat{\mathcal{O}}_x$. Let $\hat{m}_x$ denote the ideal of $\hat{\mathcal{O}}_x$ consisting of formal power series with constant term 0, and $\hat{m}_x^k$ the ideal of formal power series having no terms of degree $< k$. In view of the inclusion $\mathcal{O}_x \hookrightarrow \hat{\mathcal{O}}_x$, it follows that $\hat{m}_x^k \cap \mathcal{O}_x = m_x^k$, where m_x is the maximal ideal in $\mathcal{O}_x$. The following lemma is required for the proof of the above theorem:

Lemma 1.9 *Let $I \subset \mathcal{O}_x$ be any ideal in $\mathcal{O}_x$. Then $I\hat{\mathcal{O}}_x \cap \mathcal{O}_x = I$.*

Proof Since $\mathcal{O}_x$ is Noetherian, write $I = (f_1, \ldots, f_m)$, for $f_i \in \mathcal{O}_x$. Then $I\hat{\mathcal{O}}_x \cap \mathcal{O}_x$ is the set of power series linear combinations of $f_1, \ldots, f_m$ that belong to $\mathcal{O}_x$ also. Consider $F = a_1 f_1 + \ldots + a_m f_m$, $a_i \in \hat{\mathcal{O}}_x$, and let $a_i^{(k)}$ be the truncation of a_i to total degree $< k$. Now, suppose $F \in \mathcal{O}_x$ also, so that $F \in I\hat{\mathcal{O}}_x \cap \mathcal{O}_x$. Hence, we may rewrite F as

$$F = a_1^{(k)} f_1 + \ldots + a_m^{(k)} f_m + \varepsilon, \tag{3}$$

where ε is a $\mathbb{C}$-linear combination of monomials of total degree $\geq k$. Hence, $\varepsilon \in \hat{m}_x^k$, and since $F \in \mathcal{O}_x$, it follows from (3) that $\varepsilon \in \mathcal{O}_x$, so that $\varepsilon \in \hat{m}_x^k \cap \mathcal{O}_x = m_x^k$. So, $F \in m_x^k + I$, and therefore

$$I\hat{\mathcal{O}}_x \cap \mathcal{O}_x \subset m_x^k + I. \tag{4}$$

Now, consider the quotient ring $\mathcal{O}_x/I$, with maximal ideal

$$(m_x + I)/I = m_x/(m_x \cap I) = m_x/I.$$

From (4), it follows that any element of $(I\hat{\mathcal{O}}_x \cap \mathcal{O}_x)/I$ belongs to

$$(m_x^k + I)/I = m_x^k/(m_x^k \cap I) = (m_x/(m_x \cap I))^k = (m_x/I)^k.$$

But this is true for all k, so that

$$(I\hat{\mathcal{O}}_x \cap \mathcal{O}_x)/I \subset \bigcap_i (m_x/I)^i = 0$$

by Krull's intersection theorem. Hence, $I\hat{\mathcal{O}}_x \cap \mathcal{O}_x = I$, as required. □

Proof of Theorem 1.8 We prove the theorem for $A = \mathcal{O}_x \subset \hat{\mathcal{O}}_x$. First, we show that any non-zero non-unit is a product of irreducibles. Suppose not, and so by the Noetherian property, the class of principal ideals generated by an element which cannot be written as a product of irreducibles has a maximal element, (f). Since f is not irreducible, let $f = gh$, for g, h non-zero, non-units. Then $f \in (g) \cap (h)$, where g and h are necessarily products of irreducibles, and therefore so is f.

To show that $\mathcal{O}_x$ is a UFD, it remains to prove that all irreducible elements of $\mathcal{O}_x$ are prime; that is, it remains to show that for $f \in \mathcal{O}_x$ irreducible, if $f|ab$ then $f|a$ or $f|b$, where the division occurs in $\mathcal{O}_x$. We now use the result from theorem 1.6 that the formal power series ring $\hat{\mathcal{O}}_x$ is a UFD to write

$$f = df', \quad a = da', \tag{5}$$

where d is the highest common factor of f and a in $\hat{\mathcal{O}}_x$. The goal now is to replace a', f' with elements of $\mathcal{O}_x$. As before, define $a'^{(k)}$ and $f'^{(k)}$ to be the truncations to total degree $< k$ of a' and f', respectively. Now, consider

$$af'^{(k)} - fa'^{(k)} = a(f'^{(k)} - f') - f(a'^{(k)} - a'),$$

where the terms in parentheses are contained in $\hat{m}_x^k = m_x^k\hat{\mathcal{O}}_x$. Hence, it follows from lemma 1.9 that

$$af'^{(k)} - fa'^{(k)} \in (am_x^k\hat{\mathcal{O}}_x + fm_x^k\hat{\mathcal{O}}_x) \cap \mathcal{O}_x = am_x^k + fm_x^k.$$

Therefore, $af'^{(k)} - fa'^{(k)} = ar - fs$, for some $r, s \in m_x^k$, and so

$$a(f'^{(k)} - r) - f(a'^{(k)} - s) = 0. \tag{6}$$

Now, set $f'' = f'^{(k)} - r$ and $a'' = a'^{(k)} - s$, so that f'' and f' are equal up to terms of total degree $< k$, and similarly for a'' and a'. Substituting into (6), we obtain the equation

$$af'' = fa'' \tag{7}$$

in $\mathcal{O}_x$. Viewing this as an equation in $\hat{\mathcal{O}}_x$, we can substitute (5) into (7) to give $da'f'' = df'a''$, and so

$$a'f'' = f'a'' \tag{8}$$

(can divide by d since we are in an integral domain). From (5), we have that a' and f' are relatively prime, so due to the unique factorisation property of $\hat{\mathcal{O}}_x$, it follows from (8) that $f'|f''$ in $\hat{\mathcal{O}}_x$. Write

$$f'' = uf'. \tag{9}$$

Now, as remarked earlier, f'' and f' have the same lowest degree term, where we can choose k so that there is at least one non-zero term of degree $< k$. Hence, u must have constant term 1, so that u is invertible in $\hat{\mathcal{O}}_x$, and so we can write $f' = u^{-1}f''$ in $\hat{\mathcal{O}}_x$. Therefore,

$$f = df' = u^{-1}df'' \in f''\hat{\mathcal{O}}_x \cap \mathcal{O}_x = f''\mathcal{O}_x,$$

where again the last equality follows from lemma 1.9. This means that f'' divides f in $\mathcal{O}_x$, and we have that $u^{-1}d \in \mathcal{O}_x$. But f is irreducible in $\mathcal{O}_x$, so either $u^{-1}d$ or f'' is a unit of $\mathcal{O}_x$.

Case 1: f'' is a unit of $\mathcal{O}_x$. Then f'' is a unit of $\hat{\mathcal{O}}_x$, and so from (9), $f' = u^{-1}f''$ is a unit in $\hat{\mathcal{O}}_x$. We can then rewrite (5) as $d = (f')^{-1}f$, so that d is a multiple of f. Then $a = da'$ is a multiple of f in $\hat{\mathcal{O}}_x$, so that $a \in f\hat{\mathcal{O}}_x \cap \mathcal{O}_x = f\mathcal{O}_x$, proving that $f|a$ in $\mathcal{O}_x$, as required.

Case 2: $u^{-1}d$ is a unit of $\mathcal{O}_x$. Therefore in $\hat{\mathcal{O}}_x$, $f = u^{-1}df'' = u^{-1}duf'$, where as a product of two units, $(u^{-1}d)u$ is a unit in $\hat{\mathcal{O}}_x$. Hence, $u^{-1}du = u^{-1}ud = d$ is a unit in $\hat{\mathcal{O}}_x$, where d was defined as the highest common factor of f and a in $\hat{\mathcal{O}}_x$, so that f and a are relatively prime in $\hat{\mathcal{O}}_x$. As before, due to the unique factorisation property of $\hat{\mathcal{O}}_x$, $f|ab$ in $\mathcal{O}_x \subset \hat{\mathcal{O}}_x$ implies that $f|b$ necessarily in $\hat{\mathcal{O}}_x$. Thus, $b \in f\hat{\mathcal{O}}_x \cap \mathcal{O}_x = f\mathcal{O}_x$, and so $f|b$ in $\mathcal{O}_x$, as required. □

Two important results that follow from the fact that the local ring $\mathcal{O}_x$ is a UFD are:

Theorem 1.10 *An irreducible subvariety $Y \subset X$ of codimension 1 has a local equation in a neighbourhood of any nonsingular point $x \in X$.*

Proof See, for example, [4, chapter II, §3, theorem 1]. □

Theorem 1.11 *If X is a nonsingular variety and $\varphi : X \dashrightarrow \mathbb{P}^N$ a rational map to projective space, then the set of points at which φ is not regular has codimension ≥ 2.*

Proof See, for example, [4, chapter II, §3, theorem 3]. □

Remark 1.12 Note that if X is singular, then there may exist Weil divisors that are not Cartier, that is to say, such a divisor on X cannot be defined locally by a single equation. A variety with the property that each local ring $\mathcal{O}_x$ is a UFD is called *factorial*, so that in a factorial variety, every Weil divisor is Cartier. We have shown that all nonsingular varieties are factorial, and it is a fact that all factorial varieties are normal. It is also known that the set of singular points of a normal variety has codimension ≥ 2, hence in particular, factorial varieties are nonsingular in codimension 1.

Definition 1.13 A variety X is called ***$\mathbb{Q}$-factorial*** if every Weil divisor D on X is $\mathbb{Q}$-Cartier, that is to say, for some $m \in \mathbb{N}$, mD is Cartier.

1.2 Differential forms

Definition 1.14 Let X be a quasiprojective variety. The ***tangent space*** Θ_x at a point $x \in X$ is defined as $(m_x/m_x^2)^*$, the vector space of all linear forms on m_x/m_x^2, where m_x is the maximal ideal of the local ring $\mathcal{O}_x$ of x. The vector space m_x/m_x^2 is called the ***cotangent space*** to X at x, denoted by Θ_x^*. At a nonsingular point $x \in X$, $\dim \Theta_x = \dim \Theta_x^* = \dim X$.

We briefly recall the definition of a differential form. Let X be a nonsingular variety of dimension n. Consider the set $\Phi[X]$ of all possible functions φ associating to each point $x \in X$ a vector $\varphi(x) \in \Theta_x^*$. Since a regular function f on X defines a differential $d_x f = f - f(x) \mod m_x^2 \in \Theta_x^*$ at x, any $f \in k[X]$ gives rise to an element $\varphi \in \Phi[X]$ by defining $\varphi(x) = d_x f$, denoted by df. Write $\Phi^r[X]$ for the set of all functions sending each point $x \in X$ to an element of $\bigwedge^r \Theta_x^*$.

Definition 1.15 A ***regular differential r-form*** on X is an element $\varphi \in \Phi^r[X]$ such that for all $x \in X$, there exists a neighbourhood U such that φ on U is in the $k[U]$-submodule of $\Phi^r[U]$ generated by the elements $df_1 \wedge \ldots \wedge df_r$, with $f_1, \ldots, f_r \in k[U]$. The regular differential r-forms on X form a module over $k[X]$, denoted by $\Omega^r[X]$. Hence, such regular differential r-forms $\omega \in \Omega^r[X]$ can be written in the form

$$\omega = \sum g_{i_1 \ldots i_r} df_{i_1} \wedge \ldots \wedge df_{i_r} \tag{10}$$

in a neighbourhood of every point $x \in X$, where $g_{i_1 \ldots i_r}$ and the f_{i_j} are regular functions in a neighbourhood of x.

Now, consider an open set $U \subset X$, and a differential r-form $\omega \in \Omega^r[U]$. Define an equivalence relation on pairs (ω, U), by $(\omega, U) \sim (\omega', U')$ if $\omega = \omega'$ on $U \cap U'$. An equivalence class under this relation is called a ***rational differential*** r***-form*** on X, and $\Omega^r(X)$ denotes the set of all such forms. If $f \in k(X)$, then df defines a rational differential form on X, and we have $\Omega^0(X) = k(X)$. Also, $\Omega^r(X)$ does not change if X is replaced by an open subset, so is a birational invariant.

Since $\Omega^r(X)$ is a vector space over $k(X)$ of dimension $\binom{n}{r}$ (see, for example [4, chapter III, §5, theorem 3]), in some open subset $U \subset X$, with $u_1, \dots, u_n \in k[U]$ such that the products $du_{i_1} \wedge \dots \wedge du_{i_r}$ for $1 \leq i_1 < \dots < i_r \leq n$ form a basis of $\Omega^r[U]$ over $k[U]$, $\omega \in \Omega^r(X)$ has a unique representation

$$\sum_{1 \leq i_1 < \dots < i_r \leq n} g_{i_1 \dots i_r} du_{i_1} \wedge \dots \wedge du_{i_r},$$

where $g_{i_1 \dots i_r}$ are regular in some $U' \subset U$, that is, are rational functions on X.

For $\varphi : X \to Y$ a regular map, the pullback via φ^* of a differential r-form $\omega = \sum g_{i_1 \dots i_r} du_{i_1} \wedge \dots \wedge du_{i_r}$ is given by

$$\varphi^*(\omega) = \sum \varphi^*(g_{i_1 \dots i_r}) d(\varphi^*(u_{i_1})) \wedge \dots \wedge d(\varphi^*(u_{i_r})). \tag{11}$$

The definition is now extended for when $\varphi : X \to Y$ is a rational map. Suppose X is irreducible and that $\varphi(X)$ is dense in Y. Then, analogous with how the k-algebra homomorphism $k[Y] \to k[X]$ can be extended to a field homomorphism under these conditions, since φ is regular on some open set $U \subset X$, and any open set $V \subset Y$ has non-empty intersection with the image of φ, the pullback $\varphi^* : \Omega^r(Y) \to \Omega^r(X)$ is well-defined. The computation is exactly as in (11).

Theorem 1.16 *If X and Y are nonsingular varieties, with Y projective, and $\varphi : X \dashrightarrow Y$ a rational map such that $\varphi(X)$ is dense in Y, then*

$$\varphi^* \Omega^r[Y] \subset \Omega^r[X] \tag{12}$$

Proof By theorem 1.11, φ is regular on the open set $X \backslash Z$, where $Z \subset X$ is a closed set of codimension ≥ 2. Since $\varphi(X)$ is dense in Y, it follows that for any open set $V \subset Y$, $V \cap \varphi(X \backslash Z) \neq \emptyset$, and so $\varphi^* : \Omega^r(Y) \to \Omega^r(X)$ is well defined. Hence, for $\omega \in \Omega^r[Y]$, $\varphi^*(\omega) \in \Omega^r(X)$, and moreover, $\varphi^*(\omega)$ is regular on $X \backslash Z$. We need to prove that $\varphi^*(\omega)$ is regular on the whole of X. Write $\varphi^*(\omega)$ in some open set $U \subset X$, in the form

$$\varphi^*(\omega) = \sum g_{i_1 \dots i_r} du_{i_1} \wedge \dots \wedge du_{i_r},$$

where $u_1, \ldots, u_n$ are regular functions on U such that $du_{i_1} \wedge \ldots \wedge du_{i_r}$ is a basis for $\Omega^r[U]$ over $k[U]$, and $g_{i_1 \ldots i_r}$ are regular in the open set $U \backslash (U \cap Z) \subset U$. Now, $\operatorname{codim}_U(U \cap Z) \geq 2$, so that the set of points at which $g_{i_1 \ldots i_r}$ is not regular has codimension ≥ 2. But this set is precisely the divisor of poles of $g_{i_1 \ldots i_r}$, $\operatorname{div}_\infty(g_{i_1 \ldots i_r})$, which by definition is of codimension 1. Hence, we must have that $\operatorname{div}_\infty(g_{i_1 \ldots i_r}) = 0$, and so the functions $g_{i_1 \ldots i_r}$ have no poles on U, so are regular functions on U, as required. □

Corollary 1.17 *If two nonsingular projective varieties X and Y are birational, then the spaces $\Omega^r[X]$ and $\Omega^r[Y]$ are isomorphic. Hence, differential forms make natural birational invariants.*

Theorem 1.18 *There are no non-zero global differential forms on $\mathbb{P}^n$.*

Proof Let $x_0, \ldots, x_n$ be projective coordinates on $\mathbb{P}^n$. Now, consider $U_0 = \mathbb{A}^n \subset \mathbb{P}^n$, where U_0 has coordinates $z_1 = \frac{x_1}{x_0}, \ldots, z_n = \frac{x_n}{x_0}$, and a regular differential form on U_0 given by

$$\omega = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} dz_{i_1} \wedge \ldots \wedge dz_{i_k}, \tag{13}$$

for $a_{i_1 \ldots i_k}$ a polynomial in $z_1, \ldots, z_n$.

Let $U_1 = \mathbb{A}^n \subset \mathbb{P}^n$, having coordinates $w_0 = \frac{x_0}{x_1}, w_2 = \frac{x_2}{x_1}, \ldots, w_n = \frac{x_n}{x_1}$, with transition maps

$$w_0 = \frac{x_0}{x_1} = \frac{1}{z_1}, \quad w_i = \frac{x_i}{x_1} = \frac{z_i}{z_1} \quad \text{for } i = 2, \ldots, n$$

between the two charts. Hence,

$$z_1 = \frac{1}{w_0}, \quad z_i = w_i z_1 = \frac{w_i}{w_0} \quad \text{for } i = 2 \ldots, n.$$

From (13), we have that on U_1,

$$\omega = \sum_{1 = i_1 < i_2 < \ldots < i_k} a_{i_1 i_2 \ldots i_k}\left(\tfrac{1}{w_0}, \tfrac{w_2}{w_0}, \ldots, \tfrac{w_n}{w_0}\right) d\Big(\tfrac{1}{w_0}\Big) \wedge d\Big(\tfrac{w_{i_2}}{w_0}\Big) \wedge \ldots \wedge d\Big(\tfrac{w_{i_k}}{w_0}\Big) + \sum_{1 < i_1 < i_2 < \ldots < i_k} a_{i_1 i_2 \ldots i_k}\left(\tfrac{1}{w_0}, \tfrac{w_2}{w_0}, \ldots, \tfrac{w_n}{w_0}\right) d\Big(\tfrac{w_{i_1}}{w_0}\Big) \wedge d\Big(\tfrac{w_{i_2}}{w_0}\Big) \wedge \ldots \wedge d\Big(\tfrac{w_{i_k}}{w_0}\Big).$$

Hence, on U_1,

$$\omega = \sum_{1 = i_1 < i_2 < \ldots < i_k} -\tfrac{1}{w_0^{k+1}} a_{i_1 i_2 \ldots i_k}\left(\tfrac{1}{w_0}, \tfrac{w_2}{w_0}, \ldots, \tfrac{w_n}{w_0}\right) dw_0 \wedge \widehat{dw_1} \wedge dw_{i_2} \wedge \ldots \wedge dw_{i_k} + \sum_{1 < i_1 < i_2 < \ldots < i_k} \tfrac{1}{w_0^{k+1}} a_{i_1 i_2 \ldots i_k}\left(\tfrac{1}{w_0}, \tfrac{w_2}{w_0}, \ldots, \tfrac{w_n}{w_0}\right) \sum_{j=0}^{k} (-1)^j \, w_{i_j} \, dw_{i_0} \wedge \ldots \wedge \widehat{dw_{i_j}} \ldots \wedge dw_{i_k},$$

where $w_{i_0} = w_0$, giving a pole of at least order $k+1$ at $w_0 = 0$. Hence, ω is not regular in a neighbourhood of a point such that $x_0 = 0$ ($w_0 = 0$), and so ω is not a regular differential form on $\mathbb{P}^n$. $\square$

1.3 Linear systems

Let X be a normal projective variety. Recall that to each divisor D on X, there is an associated vector space

$$L(D) = \{f \in k(X) | D + \operatorname{div}(f) \geq 0\} \cup \{0\},$$

with $l(D) = \dim L(D)$.

Definition 1.19 Two divisors D and D' on X are said to be ***linearly equivalent*** if their difference is a principal divisor, that is,

$$D - D' = \operatorname{div}(f) \quad \text{for some } f \in k(X)^*.$$

We write $D \sim D'$.

Definition 1.20 Let D be a divisor on X. The ***complete linear system*** associated to D is the set

$$|D| = \{D' \in \operatorname{Div}(X) | D' \geq 0, D' \sim D\}.$$

The complete linear system $|D|$ is parameterised by the projective space

$$\mathbb{P}(L(D)) \cong \mathbb{P}^{l(D)-1},$$

with the parameterisation given by

$$\mathbb{P}(L(D)) \to |D|, \quad f \mod k^* \mapsto D + \operatorname{div}(f).$$

Hence, we can identify $|D|$ with $\mathbb{P}(L(D))$, so that $|D|$ has the structure of a projective space.

Definition 1.21 A ***linear system*** ϑ on X is a projective subspace of a complete linear system $|D|$, so that ϑ is a subset of $|D|$ for some fixed divisor D, and is parameterised by a linear subvariety of $\mathbb{P}(L(D)) \cong \mathbb{P}^{l(D)-1}$.

Example 1.22 Let H be the hyperplane $\{x_0 = 0\}$ in $\operatorname{Div}(\mathbb{P}^n)$, and let $r \in \mathbb{Z}_+$. Then $L(rH) = \{F_r/x_0^r | F_r \in k[x_0 \ldots, x_n]_r\}$, where $k[x_0, \ldots, x_n]_r$ is the space of homogeneous polynomials of degree r in $\mathbb{P}^n$. Hence, $l(rH) = \binom{n+r}{n}$. The complete linear system $|rH|$ consists of all hypersurfaces of degree r in $\mathbb{P}^n$, and so has dimension $\binom{n+r}{n} - 1$.

Example 1.23 Let $X \subset \mathbb{P}^n$ be a projective variety, with homogeneous ideal I_X. Recall that a form F of degree $r \in \mathbb{Z}_+$ in the coordinates of $\mathbb{P}^n$, with $F \notin I_X$, cuts out an effective divisor $F_X = \{(U_i, F/x_i^r)|0 \leq i \leq n\}$ on X, where X is covered by the affine open sets $U_i = X - \{x_i = 0\}$. If $G \notin I_X$ is any other homogeneous polynomial of degree r, then F/G is a rational function on X, and

$$F_X - G_X = \operatorname{div}(F/G),$$

so that $F_X \sim G_X$. Hence, these divisors form a linear system

$$L_X(r) = \{F_X|F \in k[x_0, \dots, x_n]_r,\ F \notin I_X\}$$

on X.

Let $\varphi : X \to Y$ be a morphism of irreducible normal projective varieties, and let $\Sigma \subset Y$ be a linear system, such that for any $D \in \Sigma$, $\varphi(X) \not\subset \operatorname{Supp} D$. This condition is satisfied, for example, if $\varphi(X)$ is dense in Y. Then the pullback $\varphi^*(D)$ of an effective divisor $D \in \Sigma$ is well defined and effective, and if $\varphi(X)$ is dense in Y, the pullback φ^* defines a homomorphism $\varphi^* : \operatorname{Div} Y \to \operatorname{Div} X$, where for a principal divisor $\operatorname{div}(f)$ on Y, $\varphi^*(\operatorname{div}(f)) = \operatorname{div}(\varphi^*(f))$ is a principal divisor on X. It follows that the set of effective divisors $\{\varphi^*(D)|D \in \Sigma\}$ is a linear system on X.

Now, suppose that $\varphi : X \dashrightarrow Y$ is a dominant rational map, regular outside a subvariety Z of codimension ≥ 2 in X. Setting $U = X - Z$, we have that for $D \in \Sigma \subset Y$, $(\varphi|_U)^*(D)$ is an effective divisor on U. Hence, for an effective Cartier divisor $D \in \Sigma$, given by $\{V_i, g_i\}$ for g_i regular on V_i, $\varphi^* g_i$ is regular in the open set $U_i = (\varphi|_U)^{-1}(V_i) \subset X$. The inclusion $U \subset X$ induces a natural map $\operatorname{Div}(X) \to \operatorname{Div}(U)$, which must be a bijection since $\operatorname{codim}_X Z \geq 2$. Hence, $(\varphi|_U)^*(D)$ extends to an effective divisor $\varphi^*(D) = \overline{(\varphi|_U)^*(D)}$ on X by taking the closure in X. In this way, we can pull a linear system back via a rational map.

Definition 1.24 Let $\Sigma \subset |D|$ be an n-dimensional linear system on X, parameterised by a projective space $\mathbb{P}(V) \subset \mathbb{P}(L(D))$, and choose a basis $f_0, \dots, f_n$ of $V \subset L(D)$. The ***rational map associated to*** Σ is the map

$$\begin{aligned} \varphi_\Sigma :\ X &\longrightarrow \mathbb{P}^n \\ x &\longmapsto (f_0(x) : \dots : f_n(x)). \end{aligned} \tag{14}$$

Clearly φ_Σ depends on the choice of basis, but two different bases give rise to maps differing only by a projective automorphism of $\mathbb{P}^n$. The map φ_Σ is defined outside of the poles and common zeros of the f_i.

Definition 1.25 The intersection of the supports of all divisors in a linear system Σ is called the set of ***base points*** of Σ. If the intersection is empty, then Σ is ***base point free***.

Definition 1.26 A divisor $\overline{D} \in \Sigma$ such that $D - \overline{D} \geq 0$ for all $D \in \Sigma$ is called a ***common*** (or ***fixed***) ***component*** of the linear system. If a linear system has no common components, then it is ***movable***.

Let the highest common divisor of given divisors $D_i = \sum k_{i,j}C_j$ for $i = 1, \ldots, n$ be the divisor

$$\operatorname{hcd}\{D_1, \ldots, D_n\} = \sum l_j C_j, \quad \text{where } l_j = \min_{1 \leq i \leq n} k_{i,j}.$$

The rational map (14) fails to be regular precisely at the points of $\bigcap \operatorname{Supp}(D_i')$, where $D_i' = \operatorname{div}(f_i) - \operatorname{hcd}\{\operatorname{div}(f_0), \ldots, \operatorname{div}(f_n)\} \geq 0$ for $i = 0, \ldots, n$. See, for example, [4, chapter III, §1, theorem 2]. The divisors D_i' can be seen as the pullbacks of the hyperplanes $\{x_i = 0\}$ under the map $\varphi_\Sigma : X \dashrightarrow \mathbb{P}^n$. To see this, let $x \notin \bigcap \operatorname{Supp} D_i'$ and suppose the divisor $\operatorname{hcd}\{\operatorname{div}(f_0), \ldots, \operatorname{div}(f_n)\}$ has local equation h in a neighbourhood U of x, where the map φ_Σ is regular and defined by

$$\varphi_\Sigma = \left(\frac{f_0}{h}, \ldots, \frac{f_n}{h}\right).$$

Then in U, $\varphi_\Sigma^*(\operatorname{div}(x_i)) = \operatorname{div} \varphi_\Sigma^*(x_i) = \operatorname{div}(x_i \circ \varphi_\Sigma) = \operatorname{div}(f_i/h)$, that is, the pullback of the hyperplane x_i is equal to D_i'.

Now, consider the linear system $L_X(1)$ defined in example 1.23, the linear system of hyperplane sections on X. The hyperplanes $\{x_0 = 0\}, \ldots, \{x_n = 0\}$ define a map on $X \hookrightarrow \mathbb{P}^n$. This map is clearly regular, as $\bigcap_{i=0}^{n} \{x_i = 0\} = \emptyset$. It is in fact an embedding.

Definition 1.27 A linear system Σ on a projective variety X is ***very ample*** if the associated rational map $\varphi_\Sigma : X \dashrightarrow \mathbb{P}^n$ is an embedding, so that φ_Σ is a morphism that maps X isomorphically onto its image $\varphi_\Sigma(X)$. Equivalently, Σ is very ample if and only if Σ separates points and tangent vectors [5, chapter II, remark 7.8.2]. A divisor D is said to be ***very ample*** if the complete linear system $|D|$ is very ample. A divisor D is said to be ***ample*** if mD is very ample for some $m \in \mathbb{Z}_+$.

Hence, very ample divisors are hyperplane sections for some embedding. Now, in the proof of proposition 6.3, we make use of the correspondence between global sections of an invertible sheaf and linear systems. For completeness, the correspondence is noted here. Let D be a divisor on X, and $\mathcal{O}_X(D)$ the corresponding invertible sheaf. Then $|D|$ is in one-to-one correspondence with the set $(\Gamma(X, \mathcal{O}_X(D)) - \{0\})/k^*$. This gives $|D|$ the structure of the set of closed points of a projective space over k, $|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$.

2 Blowups

In this chapter we recall, mainly from [4], the concept of blowing up. A birational map between nonsingular projective curves is an isomorphism - this is no longer true for higher dimensional varieties. A blowup is an example of a birational map that is not biregular, and is fundamental in birational geometry. Any birational regular map that is not an isomorphism has an *exceptional divisor.* First, we give the definition of the blowup of a point in the projective space, and then generalise this to the case when we blow up subvarieties of higher dimension. We also state a result due to Hironaka that says blowups can be used to *resolve* singularities of a rational map. We then proceed to show that the inverse image of a divisor under a blowup is reducible and consists of two components: the *strict transform* of the divisor, and an exceptional divisor, and determine with what multiplicities these components occur. We also define the *discrepancy* of the blowup, and compute this number for the case when the *centre* is a smooth subvariety. In section 2.4, we describe how to compute the self-intersection of an exceptional divisor arising from the blowup of a surface, which turns out to be negative. In later chapters, we reduce computations in higher dimensions to computations on a surface, so the self-intersection computed here will be key to such computations. Finally, we extend the notion of strict transform of a divisor to that of a cycle of higher codimension.

2.1 Blowup of a point

Let $x_0, \dots, x_n$ be homogeneous coordinates for $\mathbb{P}^n$, and $y_1, \dots, y_n$ for $\mathbb{P}^{n-1}$, and set $p = (1 : 0 : \dots : 0) \in \mathbb{P}^n$. If $x = (x_0 : \dots : x_n)$, $y = (y_1 : \dots : y_n)$, then a point $(x, y) \in \mathbb{P}^n \times \mathbb{P}^{n-1}$ is denoted by $(x_0 : \dots : x_n; y_1 : \dots : y_n)$. Consider the closed subvariety $\widetilde{\mathbb{P}^n} \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by

$$x_i y_j = y_i x_j, \quad \text{for } i, j = 1, \dots, n. \tag{15}$$

Definition 2.1 The map $\sigma : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ defined as the restriction to $\widetilde{\mathbb{P}^n}$ of the projection $\mathbb{P}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^n$ is called the ***blowup*** of $\mathbb{P}^n$, with *centre* p.

If $(x_0 : \dots : x_n) \neq p$, then it follows from the defining equations of $\widetilde{\mathbb{P}^n}$ that $(y_1 : \dots : y_n) = (x_1 : \dots : x_n)$, and so outside of this point, the mapping

$$(x_0 : \dots : x_n) \to (x_0 : \dots : x_n; x_1 : \dots : x_n)$$

is inverse to σ. However, if $(x_0 : \dots : x_n) = p$, then the defining equations are satisfied by any y_i, and so the inverse image of p is given by $\sigma^{-1}(p) = p \times \mathbb{P}^{n-1}$.

Thus, σ determines an isomorphism between $\mathbb{P}^n - p$ and $\widetilde{\mathbb{P}^n} - (p \times \mathbb{P}^{n-1})$, away from the point p, which is denoted the ***centre*** of the blowup.

Now, consider a point $(p; y_1 : \ldots : y_n) \in \sigma^{-1}(p)$, where $y_i \neq 0$ for some i. Hence, the point is contained in the open set U_i of $\widetilde{\mathbb{P}^n}$ defined by $x_0 \neq 0$, $y_i \neq 0$. Hence, in U_i we may set $x_0 = 1$, $y_i = 1$, so that the defining equations (15) are reduced to the form $x_j = x_i y_j$ for $j = 1, \ldots, n$ with $j \neq i$. Hence, U_i is isomorphic to the affine space $\mathbb{A}^n$ with coordinates $y_1, \ldots, \widehat{y_i}, x_i, \ldots, y_n$. It follows that $\widetilde{\mathbb{P}^n}$ is nonsingular and irreducible. See, for example, [4, chapter II, §4.1] for the details.

More generally, let X be a quasiprojective variety, and $p \in X$ a nonsingular point. We now construct a variety $\widetilde{X}$ and a mapping $\sigma : \widetilde{X} \to X$ analogous to the above construction. Let $u_1, \ldots, u_n$ be functions regular on the whole of X such that they form a system of local coordinates on X at p, and have p as the only common zero in X. Define a subvariety $\widetilde{X} \subset X \times \mathbb{P}^{n-1}$ consisting of points $(x; t_1 : \ldots : t_n)$, with $x \in X$ and $(t_1 : \ldots : t_n) \in \mathbb{P}^{n-1}$, such that

$$u_i(x) t_j = u_j(x) t_i \quad \text{for } i, j = 1, \ldots, n. \tag{16}$$

Definition 2.2 The map $\sigma : \widetilde{X} \to X$ obtained as the restriction to $\widetilde{X}$ of the projection $X \times \mathbb{P}^{n-1} \to X$ is called the ***local blowup*** of X, with centre p.

As in the case with the blowup of $\mathbb{P}^n$, we have that σ determines an isomorphism between $\widetilde{X} - (p \times \mathbb{P}^{n-1})$ and $X - p$. Now let $y = (p; t_1 : \ldots : t_n) \in \sigma^{-1}(p)$, where $t_i \neq 0$ for some i. Hence, we may set $s_j = \frac{t_j}{t_i}$, $j \neq i$, so that the relations (16) defining $\widetilde{X}$ become $u_j = u_i s_j$ for $j = 1, \ldots, n$ with $j \neq i$. It follows that $\widetilde{X}$ is irreducible and nonsingular; that is, $\widetilde{X} = \overline{\sigma^{-1}(X - p)}$, and $s_1 - s_1(y), \ldots, \widehat{s_i - s_i(y)}, u_i - u_i(y), \ldots, s_n - s_n(y)$ are local coordinates at $y \in \sigma^{-1}(p)$ at which $t_i \neq 0$. The local blowup is independent of the choice of local coordinates.

Now, for $X \subset \mathbb{P}^m$ an irreducible quasiprojective variety, another way to realise the local blowup $\widetilde{X}$ of X at p is as a subvariety of the blowup $\widetilde{\mathbb{P}^m}$ of $\mathbb{P}^m$ at p. Let $\sigma : \widetilde{\mathbb{P}^m} \to \mathbb{P}^m$ be the blowup of $\mathbb{P}^m$ at p, with $\overline{X} \neq \mathbb{P}^m$, and X nonsingular at p. Then the inverse image $\sigma^{-1}(X)$ of X under the blowup of $\mathbb{P}^m$ centred at p is reducible and consists of two components:

$$\sigma^{-1}(X) = (p \times \mathbb{P}^{m-1}) \cup \widetilde{X}, \tag{17}$$

where $\widetilde{X} = \overline{\sigma^{-1}(X - p)}$ is irreducible. $\widetilde{X}$ is called the ***strict transform*** of X in $\widetilde{\mathbb{P}^m}$ under the map σ. The restriction of the map σ to $\widetilde{X}$ defines a regular map $\sigma : \widetilde{X} \to X$, which is then the same as the local blowup of X at p, that is to say, it is an isomorphism in some neighbourhood U of x if $x \neq p$, and a local blowup $\sigma^{-1}(U) \to U$ with centre p if $x = p$. To see that it is indeed the same as the local blowup in a neighbourhood of p, consider a system of local coordinates $u_1, \ldots, u_m$ of $\mathbb{P}^m$ at p, with p the only common zero, such that in some neighbourhood of p, X is given by local equations $u_{n+1} = \ldots = u_m = 0$, and $u_1, \ldots, u_n$ define a system of local coordinates on X at p. We have the blowup equations $u_i t_j = u_j t_i$, and for $j = n+1, \ldots, m$, $u_j(x) = 0$ for all $x \in X$. Now, since p is the only common zero of the u_j, for all $x \in X - p$, there exists an $i \in \{1, \ldots, n\}$ such that $u_i(x) \neq 0$. Then since $u_i(x)t_j = u_j(x)t_i$, we have that $t_{n+1}(x) = \ldots = t_m(x) = 0$ for $x \neq p$. Hence, $\widetilde{X} \subset X' \subset X \times \mathbb{P}^{m-1}$, where X' is defined by

$$t_{n+1} = \ldots = t_m = 0 \tag{18}$$

and

$$u_i t_j = u_j t_i \quad \text{for } i, j = 1, \ldots, n. \tag{19}$$

Writing $\mathbb{P}^{n-1}$ for the subspace of $\mathbb{P}^{m-1}$ defined by (18), we have that $X' \subset X \times \mathbb{P}^{n-1}$, defined by equations (19). Hence X' is the local blowup, so $X' = \overline{\sigma^{-1}(X - p)}$. Therefore $\widetilde{X} = X'$, as required.

As a consequence, we now have the following definition.

Definition 2.3 Let $X \subset \mathbb{P}^m$ be a quasiprojective variety, p a nonsingular point of X, and $\widetilde{X}$ the variety defined in (17). Then $\sigma : \widetilde{X} \to X$ is called the ***blowup*** of X, with centre p, where $\widetilde{X}$ is irreducible if X is, and $\sigma^{-1}(p) \cong p \times \mathbb{P}^{m-1}$, with $\widetilde{X}$ nonsingular at every point in $\sigma^{-1}(p)$.

The blowup is an isomorphism for X a curve, but in higher dimensions is a birational morphism, and only fails to be an isomorphism because the rational map σ^{-1} is not regular at p. The blowup contracts a subvariety of codimension 1 in $\widetilde{X}$ to the point p in X, with $\operatorname{codim}_X\{p\} \geq 2$. Such a subvariety is called ***exceptional***. More generally, we have the following definition.

Definition 2.4 Let $f : X \to Y$ be a birational morphism. A subvariety $Z \subset X$ is ***f-exceptional***, or exceptional for f, if $\operatorname{codim}_Y f(Z) > \operatorname{codim}_X Z$.

The same property of the blowup holds in general: if f in the above definition is not biregular, then f has an exceptional subvariety.

2.2 Blowup when the centre is a smooth subvariety

Definition 2.5 Let $U \subset X$ be a dense open set. The ***graph*** of a rational map $f : X \dashrightarrow Y$ which is regular on the set U is defined as follows:

$$\Gamma_f = \overline{\{(x, f(x))|x \in U\}} \subset X \times Y.$$

Let B be a smooth, irreducible subvariety of a quasiprojective variety $X \subset \mathbb{P}^m$, with homogeneous (radical) ideal $I(B)$, generated by homogeneous polynomials $F_1, \ldots, F_k$. Consider the graph $X(B)$ of the rational map

$$F : X \dashrightarrow \mathbb{P}^{k-1}$$

where

$$F(x) = (F_1(x) : \ldots : F_k(x)).$$

Then the blowup of X along the ideal $I(B)$ is the natural projection

$$\begin{array}{rcl} \sigma : X(B) \subset X \times \mathbb{P}^{k-1} & \longrightarrow & X \\ (x, F(x)) & \longmapsto & x. \end{array}$$

Clearly, this defines an isomorphism between $X(B) - \sigma^{-1}(B)$ and $X - B$, since the inverse map is well defined on $X - B$ as the F_i do not all vanish anywhere on $X - B$.

Definition 2.6 The ***blowup of*** X ***along the subvariety*** B is the blowup along the radical ideal $I(B)$, and is independent of the choice of generators. The inverse image $\sigma^{-1}(B)$ is called the exceptional divisor.

We can also view the blowup of an n-dimensional variety X along a smooth subvariety B of codimension k as a local blowup: let $t_1, \ldots, t_k$ be coordinates on $\mathbb{P}^{k-1}$, and choose local coordinates $u_1, \ldots, u_n$ on X at a generic point $p \in B$, so that B is defined locally by equations $u_1 = \ldots = u_k = 0$, $u_{k+1}, \ldots, u_n$ are local coordinates on B, such that the blowup $\widetilde{X} = X(B)$ is defined by equations $u_i = s_i$ for some $i \in \{1, \ldots, k\}$, $u_j = s_j s_i$ for $j = 1, \ldots, k$ with $j \neq i$, and $u_j = s_j$ for $j = k+1, \ldots, n$ in the open set $t_i \neq 0$, where $s_j = \frac{t_j}{t_i}$ for $j = 1, \ldots, k$ with $j \neq i$. Set $E = \sigma^{-1}(B)$. A local system of coordinates at any point $y \in \sigma^{-1}(p) \subset E$ is given by $s_1 - s_1(y), \ldots, s_i, \ldots, s_k - s_k(y), s_{k+1}, \ldots, s_n$, and the local equation of E is $s_i = 0$.

As an application as to the importance of blowups, we have the following theorem.

Theorem 2.7 *Let $\varphi : X \dashrightarrow Y$ be a rational map of projective varieties. Then there exists a chain of blowups*

$$X_m \overset{\sigma_m}{\to} X_{m-1} \to \ldots \to X_{i+1} \overset{\sigma_{i+1}}{\to} X_i \to \ldots \to X_1 \overset{\sigma_1}{\to} X$$

such that the composite $\psi = \varphi \circ \sigma_1 \circ \ldots \circ \sigma_m : X_m \to Y$ is regular.

Such a process is called a resolution of singularities of the map φ, and its existence in arbitrary dimension (in characteristic zero) was first proved by Hironaka [26]. For the case when Y is a singular variety (over a field of characteristic zero), Hironaka proved that Y has a resolution of singularities, that is, there exists a nonsingular variety X with a proper birational morphism from X to Y; in actual fact, Hironaka shows that the resolution may be chosen to be isomorphic outside the singular locus.

2.3 Strict transform of divisors passing through the centre

Consider the (local) blowup $\sigma : \widetilde{X} \to X$ of X along a smooth subvariety B as described above, with $\operatorname{codim}_X B \geq 2$, and let D be a prime divisor on X, with $B \subset \operatorname{Supp} D$. Now, analogous to (17), we have that the inverse image $\sigma^{-1}(D)$ is reducible and consists of two components: the exceptional divisor $E = \sigma^{-1}(B)$, and the strict transform $\widetilde{D}$ of D, defined as the closure in $\widetilde{X}$ of $\sigma^{-1}(D - B)$. In terms of divisors,

$$\sigma^*(D) = \widetilde{D} + mE,$$

where $\widetilde{D}$ occurs with coefficient 1 since $\sigma : \widetilde{X} - E \to X - B$ is an isomorphism. We now show that $m = \operatorname{mult}_B D$. Suppose that $\operatorname{mult}_B D = r$. Then for the local equation f of D in a neighbourhood of a generic point $p \in B$, we have that $f \in m_B^r \setminus m_B^{r+1}$. We also have that $\sigma^*(D)$ has local equation $\sigma^*(f)$ in a neighbourhood of any point $y \in \sigma^{-1}(p)$. Let

$$f = \varphi(u_1, \ldots, u_k) + \psi,$$

with φ a form of degree r, and $\psi \in m_B^{r+1}$. Hence,

$$(\sigma^*(f))(s_1, \ldots, s_k) = \varphi(s_1 s_i, \ldots, s_i, \ldots, s_k s_i) + \sigma^*(\psi).$$

Then $\psi \in m_B^{r+1}$ means that we can write $\psi = F(u_1, \ldots, u_k)$, with F a form of degree $r+1$ in $u_1, \ldots, u_k$, with coefficients in $\mathcal{O}_B$. It follows that $\sigma^*(\psi) = (\sigma^*(F))(s_1 s_i, \ldots, s_i, \ldots, s_k s_i)$, and so

$$(\sigma^*(f))(s_1, \ldots, s_k) = s_i^r \left(\varphi(s_1, \ldots, 1, \ldots, s_k) + s_i(\sigma^*(F))(s_1, \ldots, 1, \ldots, s_k)\right).$$

Since $\varphi(s_1, \ldots, 1, \ldots, s_k)$ is not divisible by s_i, it follows that $m = r$, as required.

Definition 2.8 Let $\sigma : \widetilde{X} \to X$ be the blowup of X at a smooth subvariety B. Then we have that

$$K_{\widetilde{X}} = K_X + a(E)E,$$

where E is the exceptional divisor of the blowup, and the coefficient $a(E)$ is called the ***discrepancy*** of the blowup.

Proposition 2.9 *The discrepancy of the blowup $\sigma : \widetilde{X} \to X$ at a smooth subvariety B of* codimension k *in an n-dimensional variety X is given by $\delta = \operatorname{codim} B - 1$.*

Proof Choose local coordinates $u_1, \dots, u_n$ on X at a generic point in B, so that B is defined locally by equations $u_1 = \dots = u_k = 0$, such that the blowup $\widetilde{X}$ is defined by equations $u_1 = s_1$, $u_j = s_j s_1$ for $j = 2, \dots, k$, and $u_j = s_j$ for $j = k+1, \dots, n$ in the open set $t_1 \neq 0$. Then the exceptional divisor E of the blowup is given by $s_1 = 0$. Now, take a form $\omega = du_1 \wedge \dots \wedge du_n$ on X. Then

$$\begin{aligned}
\sigma^*(\omega) &= d(\sigma^*(u_1)) \wedge \dots \wedge d(\sigma^*(u_k)) \wedge d(\sigma^*(u_{k+1})) \wedge \dots \wedge d(\sigma^*(u_n)) \\
&= ds_1 \wedge d(s_2 s_1) \wedge \dots \wedge d(s_k s_1) \wedge ds_{k+1} \wedge \dots \wedge ds_n \\
&= ds_1 \wedge (s_1 ds_2 + s_2 ds_1) \wedge \dots \wedge (s_1 ds_k + s_k ds_1) \wedge ds_{k+1} \wedge \dots \wedge ds_n \\
&= s_1^{k-1} ds_1 \wedge \dots \wedge ds_n.
\end{aligned}$$

Hence, $\operatorname{div}(\sigma^*(\omega)) = \operatorname{div}(s_1^{k-1}) = (k-1)\operatorname{div} s_1 = (k-1)E$. That is to say, $\delta = \operatorname{codim} B - 1$. □

2.4 Intersection theory on blowups

In this section and the next, we explain how to compute some basic intersection numbers on blowups. From the latter section, we will only need the numbers - for the complete intersection theory, which we also use here, see [6]. We first recall the following two properties of a birational regular map $f : Y \to X$ between nonsingular projective surfaces.

(i) For D_1 and D_2 divisors on X, $f^*(D_1).f^*(D_2) = D_1.D_2$;

(ii) For E a divisor on Y, all of whose components are exceptional, $f^*(D).E = 0$ for any divisor D on X.

Now, let $\sigma : \widetilde{X} \to X$ be the blowup of a surface X at a smooth point $p \in X$. That is to say, for t_1, t_2 coordinates on $\mathbb{P}^1$, u_1, u_2 local coordinates on X at p, the blowup in a neighbourhood of p is given by $u_1 t_2 = u_2 t_1$, and in the open set $t_1 \neq 0$ by

$$u_1 = s_1 \quad \text{and} \quad u_2 = s_2 s_1, \tag{20}$$

where $s_2 = \frac{t_2}{t_1}$. Let $E = \sigma^{-1}(p)$, and for any point $y \in E$, a system of local coordinates at y on E is given by s_1 and $s_2 - s_2(y)$, with the local equation of E being $s_1 = 0$. Let D be a divisor on X, with local equation $u_2 = 0$. It follows from section 2.3 that $\sigma^*(D) = \widetilde{D} + E$, and from (20) that $\widetilde{D}$ is given by local equation $s_2 = 0$. Hence, the intersection multiplicity $\widetilde{D}.E$ is 1. It follows that

$$1 = \widetilde{D}.E = (\sigma^*(D) - E).E = \sigma^*(D).E - E^2 = -E^2.$$

That is to say, $E^2 = -1$.

In chapter 6, we will make use of the following lemma.

Proposition 2.10 *Let $f : Y \to X$ be a regular birational map, with $\dim X = n$. If E is an effective divisor on Y, all of whose components are exceptional, then there is no other effective divisor D that is linearly equivalent to E.*

Proof Let H be the class of a hyperplane in X, and suppose $D \sim E$ for an effective divisor D on Y. Then $(f^*(H))^{n-1}.E = (f^*(H))^{n-1}.D = 0$, and so D is exceptional: if not, then $\operatorname{codim}_X f(D) = 1$, and so $H^{n-1}.f(D) > 0$ giving $(f^*(H))^{n-1}.D > 0$. Hence, D is supported on the same set of Weil divisors as E, which is finite as there are only finitely many exceptional divisors. This cannot happen in a linear system of positive dimension. □

2.5 Strict transform of cycles of higher codimension

Lemma 2.11 *Let $\sigma : \widetilde{X} \to X$ be the blowup of X at a point $x \in X$. Then for Y a cycle of codimension $k \geq 2$ on X,*

$$\widetilde{Y} = \sigma^*(Y) - \operatorname{mult}_x Y(H_E^{k-1}),$$

for $\widetilde{Y}$ the strict transform of Y defined as the closure in $\widetilde{X}$ of $\sigma^{-1}(Y - x)$, and H_E the class of a hyperplane section of the exceptional divisor of the blowup, where H_E^{k-1} is the self intersection inside E.

Proof Since H_E is the class of a hyperplane section of $E \cong \mathbb{P}^{n-1}$, we have that H_E^{k-1} is of codimension k in $\widetilde{X}$. It follows from the general theory of cycles that we can write

$$\widetilde{Y} = \sigma^*(Y) - r(H_E^{k-1}). \tag{21}$$

Now suppose that $\operatorname{mult}_x Y = m$; we show that $r = m$. Let R be a generic cycle on X, smooth at x, such that x is an isolated point of the intersection $Y \cap R$, the intersection is 0-dimensional, and the tangent space $T_x R$ is of general position. Then we have that the local intersection multiplicity $(Y.R)_x$ is equal to m. It follows that

$$\sigma^*(Y).\widetilde{R} = (Y.R)_x + \sum_{z \neq x} (Y.R)_z,$$

and from (21) we have that

$$\widetilde{Y}.\widetilde{R} + r(H_E^{k-1}).\widetilde{R} = (Y.R)_x + \sum_{z \neq x} (Y.R)_z.$$

By construction, the intersection $\widetilde{Y} \cap \widetilde{R}$ on E is empty, so that $(\widetilde{Y}.\widetilde{R})_E = 0$. Hence, on E

$$r(H_E^{k-1}).\widetilde{R} = (Y.R)_x = m.$$

Furthermore, since H_E is just a hyperplane in $\mathbb{P}^{n-1}$, it follows that $(H_E^{k-1}).\widetilde{R} = 1$, where the intersection is in E. Hence, $r = \operatorname{mult}_x Y$, as required. □

3 Rationally connected varieties

In the classical setting, the rationality problem is the problem of explicitly solving a system of algebraic equations. Informally, if a variety admits a parameterisation by a projective space, then it is *rational.* In the first section, we define precisely what it means for a variety to be *rational*, *unirational*, *rationally connected*, *rationally chain connected*, *ruled* and *uniruled*, and state the relationship between such varieties. In particular, we recall existing results in birational geometry as to whether any of these notions coincide. We give the definition of a Fano variety, which will be the class of varieties we consider for the rest of the thesis. Rationally connected fibre spaces, and structures of a rationally connected fibre space are defined, and we discuss the implications for a variety with structures. In the final section of this chapter, we consider the rationality problem, and give a brief history of the work in this area of birational geometry.

3.1 Rational and rationally connected

Definition 3.1 Let X be a variety. Then X is

(i) ***rational*** if X is birational to $\mathbb{P}^n$, for some n.

(ii) ***unirational*** if there is a dominant rational map $\phi : \mathbb{P}^n \dashrightarrow X$, for some n.

(iii) ***rationally connected*** if for any two generic points x and y of X, there exists an irreducible rational curve connecting x and y. In other words, there exists a morphism

$$f : \mathbb{P}^1 \to X,$$

such that $f(0) = x$ and $f(\infty) = y$.

(iv) ***rationally chain connected*** if for any two generic points x and y of X, there exists a chain C of rational curves C_i connecting x and y, so that

$$C = \bigcup_{i=0}^{k} C_i,$$

where $x \in C_0$, $y \in C_k$, and $C_i \cap C_{i+1} \neq \emptyset$. In other words, there are morphisms

$$f_i : \mathbb{P}^1 \to X, \quad 0 \leq i \leq k,$$

such that $f_0(0) = x$, $f_i(\infty) = f_{i+1}(0)$ for $0 \leq i \leq k-1$, and $f_k(\infty) = y$.

(v) ***ruled*** if X is birational to $Y \times \mathbb{P}^1$, where $\dim Y = \dim X - 1$.

(vi) ***uniruled*** if there is a dominant rational map $\phi : Y \times \mathbb{P}^1 \dashrightarrow X$, where $\dim Y = \dim X - 1$.

Remark 3.2 (i) Since any two points in $\mathbb{P}^n$ lie on a line, it follows that $\mathbb{P}^n$ is rationally connected. Furthermore, if X is rationally connected, and $\varphi : X \dashrightarrow Y$ is a dominant rational map, then Y is also rationally connected: if $y_1, y_2 \in Y$ are generic points in the image of φ, then choose generic points $x_i \in \varphi^{-1}(y_i)$ on X. By assumption, $x_1, x_2 \in X$ can be joined by a rational curve $f : \mathbb{P}^1 \to X$, and so y_1 and y_2 are joined by the rational curve $\varphi \circ f : \mathbb{P}^1 \to Y$. Hence, every unirational variety is rationally connected. We therefore have that rational implies unirational, unirational implies rationally connected, rationally connected implies rationally chain connected, and it is also true that rationally chain connected implies uniruled. (ii) If X is rationally connected, then it follows from the standard technique of deformation theory [8, chapter IV] that for every integer $m > 0$, and every $x_1, \ldots, x_m \in X$, there is an irreducible rational curve through $x_1, \ldots, x_m$.

Example 3.3 A smooth irreducible quadric hypersurface $X_2 \subset \mathbb{P}^n$ defined by a homogeneous equation of degree 2 is rational: project X_2 from any of its points to give a birational map between X_2 and $\mathbb{P}^{n-1}$.

Now, in light of theorem 1.16, corollary 1.17, and theorem 1.18, it follows that rational varieties have no non-zero global differential forms, and in fact, no non-zero global covariant tensors; in particular, all pluri-canonical linear systems are empty. The same is true for unirational varieties in characteristic zero. It is therefore natural to consider whether these notions coincide; this question is called the Lüroth problem. A curve C is rational if and only if its genus, $g_C = \dim H^0(C, \Omega^1_C)$, is zero, and so it follows from what we just said that there is no distinction between rational and unirational for curves. In characteristic zero, it follows from the Castelnuovo rationality criterion that every unirational surface is rational, and so again the notions coincide.

For higher dimensional varieties, there are plenty of examples of unirational varieties that are not rational. For example, in [9], Iskovskikh and Manin proved that any smooth quartic threefold in $\mathbb{P}^4$ is not rational, whereas in [10], Segre found examples of smooth unirational quartics, therefore giving a negative solution to the three-dimensional Lüroth problem. Another example of a variety that is unirational, but not rational, is that of a smooth cubic threefold in $\mathbb{P}^4$ [11]. Whether or not a smooth cubic hypersurface $Q \subset \mathbb{P}^n$ is rational for $n \geq 5$ is an open problem.

Example 3.4 For a smooth hypersurface $X_d \subset \mathbb{P}^n$ of degree $d > n$, $\dim H^0(X, \Omega_X^{n-1}) = \binom{d-1}{n}$. Hence, X is not rational.

We have seen that not every unirational variety is rational, and it is natural to consider other classes of varieties that are close to being rational in some sense. It is a fact that a rationally connected variety X has no differential forms, that is to say, $\dim H^0(X, \Omega_X^p) = 0$ for all $p > 0$, so these varieties are similar to rational varieties in this respect. It is a conjecture that rationally connected does not imply unirational. For example, every smooth quartic $X \subset \mathbb{P}^4$ is rationally connected, but it is conjectured that a general smooth three-dimensional quartic is not unirational. Whereas the above example shows that there exist many smooth projective varieties that are not rational, it is more difficult to find examples of non-rational varieties that have no global differential forms.

Theorem 3.5 **(Kollár-Miayoka-Mori [7])** *Let X be a smooth projective variety. Then*

$$-K_X \text{ ample} \implies X \text{ is rationally connected.}$$

Example 3.6 A smooth hypersurface $X_d \subset \mathbb{P}^n$ of degree d has canonical class

$$K_{X_d} = (K_{\mathbb{P}^n} + X_d)|_{X_d} = -(n+1)H + dH = (d-n-1)H,$$

for H the class of a hyperplane section of X_d. Hence, X_d is rationally connected for $d \leq n$.

Definition 3.7 A smooth projective variety X is called a ***Fano variety*** if its anticanonical divisor $-K_X$ is ample. If X is a normal projective variety, then X is said to be a (terminal) ***$\mathbb{Q}$-Fano variety*** if X has only terminal singularities (see remark 4.23 for the definition) and some positive integral multiple $-nK_X$, $n \in \mathbb{N}$, of an anticanonical Weil divisor $-K_X$ is an ample Cartier divisor.

In dimension 1, there is a unique Fano variety up to isomorphism, the projective line $\mathbb{P}^1$. Fano varieties of dimension 2 are called del Pezzo surfaces, and are again all rational. The example already discussed of a smooth quartic $V_4 \subset \mathbb{P}^4$ is a Fano variety of dimension 3 that is non-rational. In example 3.6, we saw that smooth hypersurfaces $X_d \subset \mathbb{P}^n$ of degree $d \leq n$ are Fano, and it follows from theorem 3.5 that all smooth Fano varieties are rationally connected.

3.2 Rationally connected fibre spaces

Definition 3.8 A surjective morphism $\pi : X \to S$ of projective varieties is called a ***rationally connected fibre space*** if the base S and the generic fibre $\pi^{-1}(s)$ over $s \in S$ are rationally connected varieties. A ***section*** of π is a morphism $\sigma : S \to X$ such that $\sigma \circ \pi$ is the identity.

In 2001, Graber, Harris, and Starr proved the following theorem.

Theorem 3.9 **(Graber-Harris-Starr [27])** *Let $\pi : X \to \mathbb{P}^1$ be a morphism of varieties over an algebraically closed field of characteristic zero, whose generic fibre is rationally connected. Then π has a section.*

The following corollary follows immediately from [7] and [8] (alternatively, see [27, corollary 1.3]).

Corollary 3.10 *Let $\pi : X \to B$ be a morphism of varieties over an algebraically closed field of characteristic zero. If both the generic fibre of π, and B, are rationally connected, then X is also rationally connected.*

Hence, if $\pi : X \to S$ is a rationally connected fibre space, then the variety X is itself rationally connected by corollary 3.10. Trivial cases occur when $X = S$ or when S is just a point, and so we only consider non-trivial fibre spaces.

Definition 3.11 A ***structure of a rationally connected fibre space*** on a variety X is defined as a birational map $\chi : X \dashrightarrow X^+$ onto a variety X^+, with a fixed morphism $\pi^+ : X^+ \to S^+$ which is a rationally connected fibre space. It follows from what we said after definition 3.1 that this is equivalent to a rational map $\varphi : X \dashrightarrow S$ with rationally connected generic fibre.

Denote by $RC(X)$ the set of distinct non-trivial structures of a rationally connected fibre space, where two structures $\varphi_1 : X \dashrightarrow S_1$ and $\varphi_2 : X \dashrightarrow S_2$ are considered the same if there is a birational map $\psi : S_1 \dashrightarrow S_2$, such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, leftrightarrow, "\mathrm{id}"] \arrow[d, "\varphi_1"'] & X \arrow[d, "\varphi_2"] \\ S_1 \arrow[r, dashed, "\psi"'] & S_2 \end{tikzcd}$$

That is to say, $\varphi_2 = \psi \circ \varphi_1$, so that φ_1 and φ_2 have the same generic fibre. Now, suppose that for an algebraic variety X, there is a morphism $\pi : X \to S$ of X

onto a variety S of smaller positive dimension. Then a generic fibre has dimension $\dim X - \dim S$, and so we can view X as a family of varieties $\pi^{-1}(s)$, $s \in S$, parameterised by the base S. The geometry of the variety X is therefore reduced to that of a generic fibre and the degree of 'twistedness' of the family of fibres over the base. From the opposite viewpoint, we can construct new varieties of dimension $\dim S + \dim F$ as fibrations for a given base S and fibre F, so that varieties of arbitrary dimension can be constructed in this way. It is therefore clear why we should want to study the structures of a rationally connected fibre space on a variety: rationally connected varieties with no non-trivial structures of a rationally connected fibre space do not arise from varieties of smaller dimension in the sense just described, so represent a new classification type. If the variety admits only finitely many structures, then there are only finitely many ways in which the reduction in dimension can occur.

By definition, the set of rationally connected structures $RC(X)$ on X is a birational invariant of X, which makes it possible to use it for the purposes of birational classification. For instance, if $RC(X)$ is empty, then X cannot be rational.

In chapters 5 and 6, we prove *birational superrigidity* of one class of (rationally connected) singular primitive Fano varieties V, that is, with $\operatorname{Pic} V = \mathbb{Z}K_V$. The technical definition of superrigidity is given in chapter 4.1, and an immediate consequence is that there are no non-trivial structures of a rationally connected fibre space on V, so that $RC(V) = \emptyset$, and that the variety is non-rational.

3.3 The rationality problem

The rationality problem, that is, the problem of identifying which varieties are rational, goes back to the beginning of algebraic geometry. By their very definition, rational varieties are the 'simplest' kind of algebraic varieties, so that it is natural to want to classify varieties according to this criterion. The rationality problem is typically very hard, or very easy. The latter alternative includes the case when there is an explicit geometric construction, such as using stereographic projection to construct a birational map between a smooth irreducible quadric hypersurface and the projective space (example 3.3), and also when there is a birational invariant that can be used to rule out rationality immediately, such as existence of non-zero global differential forms on the variety. Obviously, since there are no non-zero global differential forms on the projective space (which is rational), this birational invariant cannot be used to solve the rationality problem for varieties with likewise no forms, such as a smooth three-dimensional quartic in $\mathbb{P}^4$ (or indeed any rationally connected variety).

Describing the group $\mathrm{Bir}(V)$ of birational self-maps of a variety V is a closely connected problem. A famous result in two-dimensional birational geometry is due to M. Noether (around 1870), which states that the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ of birational automorphisms of the projective plane is generated by the subgroup $\mathrm{Aut}(\mathbb{P}^2)$ of projective automorphisms and any standard quadratic transformation [17]. In fact, Noether's arguments were incomplete, and it would take another 30 years for a complete proof.

In the early part of the last century, Gino Fano attempted to extend two dimensional birational methods to threefolds [14, 15]. His object of study was a quartic threefold in $\mathbb{P}^4$, and he started by trying to describe the birational transformations of such a variety. Though many of his arguments were later shown to be wrong or unsubstantiated, one of his most impressive claims was what was later proved in the ground breaking paper of Iskovskikh and Manin [9] (1971). We earlier recalled this result, that is, of the non-rationality of a smooth quartic $V \subset \mathbb{P}^4$. In fact, they showed that the groups $\mathrm{Bir}(V)$ and $\mathrm{Aut}(V)$ of birational and biregular automorphisms, respectively, coincide. In [16], Matsumura and Monsky showed that for $X \subset \mathbb{P}^n$ a nonsingular hypersurface of degree d, the group $\mathrm{Aut}(X)$ is finite for $n \geq 3$, $d \geq 3$ (except for the case $n = 3$, $d = 4$). It therefore follows that $\mathrm{Bir}(V)$ is finite, whereas $\mathrm{Bir}(\mathbb{P}^n)$ is infinite, and so V is not rational. The procedure used in [9] is called the *method of maximal singularities*, which we describe in chapter 4, and use in chapter 5.

In the 1970's and 1980's, Iskovskikh and his students A.A. Zagorskii, V.G. Sarkisov [18, 19], S.L. Tregub [20, 21], S.I. Khashin [22], and A.V. Pukhlikov [23, 32, 40, 24] published results in this field. In what follows, we give a brief survey of some of these and more recent results on birational rigidity. After the Iskovskikh and Manin result on the three-dimensional quartic, it was conjectured that a smooth hypersurface

$$V_M \subset \mathbb{P}^M$$

of degree $M \geq 5$ (and index 1) is birationally superrigid, generalising the aforementioned result. The case of the four-dimensional quintics, $M = 5$, was proved by Pukhlikov in [23] (1987). In the nineties, Pukhlikov developed a new version of the method of maximal singularities which was presented in [25] (1995), [52] (2000), and [42] (1998). The re-worked method, together with the technique of *hypertangent divisors*, led to a proof of superrigidity of generic hypersurfaces of degree $M \geq 6$ in [42]

(1998). Cheltsov gave an alternative proof of superrigidity of the four-dimensional quintics in [49] (2000). He also claimed superrigidity of any smooth hypersurface for $M = 6, 7, 8$, but as it was discovered later, his argument was faulty and a complete proof was obtained only in [53] (2008); see also [54] (2010). Some new ideas aimed at proving superrigidity for arbitrary smooth hypersurfaces (based on the method of maximal singularities, combined with the connectedness principle of Shokurov and Kollár [12, 38]) were introduced in [41] (2002). For a discussion of the above results on the birational rigidity of smooth hypersurfaces of degree M in $\mathbb{P}^M$, $M \geq 4$, see the survey paper [51].

The effect of introducing singularities was also considered: in 1989, Pukhlikov proved that a three-dimensional quartic $V_4 \subset \mathbb{P}^4$ with an isolated non-degenerate double point is birationally rigid [40]. The proof was later simplified by Corti in [39] (2000) by applying the connectedness principle of Shokurov and Kollár [12, 38]. The result was then generalised to higher dimensions: in [45] (2002) it was proved that a general Fano hypersurface $V = V_M \subset \mathbb{P}^M$ of index 1 with isolated singularities in general position is birationally rigid.

In 2001, Pukhlikov proved that generic Fano complete intersections

$$V = V_{d_1 \cdot \ldots \cdot d_k} = F_1 \cap \cdots \cap F_k \subset \mathbb{P}^{M+k},$$

for $F_i \subset \mathbb{P}^{M+k}$ a hypersurface of degree $d_i \geq 2$ and integers $d_1, \ldots, d_k$ satisfying the relation $d_1 + \ldots + d_k = M + k$ (so that V is of index 1), are birationally superrigid for $M \geq 2k + 1$, $k \geq 2$ [43]. The birational superrigidity of a smooth four-dimensional complete intersection $V_8 = F_2 \cap F_4$ of a quadric and quartic in $\mathbb{P}^6$ that contains no two-dimensional linear subspace of $\mathbb{P}^6$ was proved by Cheltsov in [50] (2003).

A generic Fano double cover

$$\sigma : V \to Q_m \subset \mathbb{P}^{M+1},$$

where Q is a hypersurface of degree $m \geq 3$, and the map σ is ramified over the smooth divisor $W = W_{m \cdot 2l} = W_{2l}^* \cap Q$ for $W_{2l}^* \subset \mathbb{P}^{M+1}$ a hypersurface of degree $2l$, $m + l = M + 1 \geq 5$, was proved to be birationally superrigid in [44] (2000). Birational rigidity for the case of a smooth double space ($m = 1$) and smooth double quadric ($m = 2$) was already proved in [32] (1989), without assuming them to be generic. Iterating the procedure of making a double cover over a given variety produces large families of smooth higher-dimensional Fano varieties that generalise Fano double hypersurfaces, and are realised as complete intersections in weighted projective spaces. A generic variety of these families is birationally rigid - see [46] (2003) for a precise formulation and proof.

The birational rigidity of $\mathbb{Q}$-Fano threefold hypersurfaces of index 1 in weighted projective spaces was shown in [47] (2000). In a paper by Corti and Mella [48] (2004), they prove birational rigidity of a specific class of singular quartic threefolds, that is, for $X = X_4 \subset \mathbb{P}^4$ a quartic threefold, with a singularity $P \in X$ analytically equivalent to $xy + z^3 + t^3 = 0$, but otherwise general (so nonsingular outside P). The equation of X can be written as

$$x_0^2 x_1 x_2 + x_0 a_3 + b_4 = 0,$$

where a_3 and b_4 are homogeneous forms of degree 3 and 4, respectively, in the variables $x_1, \ldots, x_4$ (where X has the required singularity if $a_3(0,0,x_3,x_4) = 0$ has three distinct roots). Moreover, they prove that for Y a Fano threefold birational to X, then either Y is biregular to X, or Y is biregular to the quasi-smooth complete intersection $Y_{3,4}$ of a quartic and cubic in the weighted projective space $\mathbb{P}(1,1,1,1,2,2)$, given by equations

$$y_1 y_2 + b_4(x_1, x_2, x_3, x_4) = 0 \quad \text{and} \quad y_1 x_1 + y_2 x_2 + a_3(x_1, x_2, x_3, x_4) = 0.$$

In [13] (2005), the birational superrigidity of direct products $V = F_1 \times \cdots \times F_K$ of primitive Fano varieties was proved, using the connectedness principle [12, 38] and technique of hypertangent divisors, where either $F_i \subset \mathbb{P}^M$ is a general hypersurface of degree M, for $M \geq 6$, or $F_i \xrightarrow{\sigma} \mathbb{P}^M$ is a general double space of index 1, for $M \geq 3$. It is then shown that every structure of a rationally connected fibre space on V is given by the projection onto a direct factor.

In example 5.4, we give some known examples of birationally superrigid varieties that are closely related to the main result of this thesis.

4 The method of maximal singularities

The *method of maximal singularities* is the main tool in use today for studying the birational geometry of Fano varieties. In the first section, we define the *threshold of canonical adjunction* and *virtual threshold of canonical adjunction* of a linear system on a variety, and use these values to define the property of *birational (super)rigidity* of a variety. In the next section, we define when a geometric discrete valuation is a *maximal singularity* of a linear system, and describe when a linear system has a maximal singularity. We then describe a process consisting of a sequence of blowups known as the *resolution of the discrete valuation*, and define an oriented graph structure on the set of exceptional divisors arising from the resolution. Finally, we describe the steps involved in the method of maximal singularities: *excluding* and *untwisting*, and give an example for which we can easily exclude certain types of maximal singularities.

4.1 Birational rigidity

We recall the basic definitions about birational superrigidity from [31]. Let X be a terminal $\mathbb{Q}$-factorial Fano variety unless otherwise stated.

Definition 4.1 Let D be a divisor on X. The ***Iitaka dimension*** of D is the largest dimension of the image of X in $\mathbb{P}^n$ by the rational map determined by the linear system $|mD|$,

$$\kappa(X, D) = \max_{m\geq 1} \dim \varphi_{|mD|}(X),$$

where $\varphi_{|mD|} : X \dashrightarrow \mathbb{P}^n$. The ***Kodaira dimension*** of X is the Iitaka dimension of the canonical divisor, $\kappa(X) = \kappa(X, K_X)$, where $\kappa(X) = -\infty$ if $|mK_X| = \emptyset$ for all $m \geq 1$.

For varieties of dimension n, it is known that κ can take on every value from 0 to n, and $-\infty$. A variety of ***general type*** is one of maximal Kodaira dimension, that is, $\kappa(X) = \dim X$.

Definition 4.2 The ***threshold of canonical adjunction*** of a divisor D on X is the number

$$c(D, X) = \sup \{b/a \mid b, a \in \mathbb{Z}_+ \setminus \{0\}, |aD + bK_X| \neq \emptyset\}.$$

If D is contained in a linear system Σ on X, then we set $c(\Sigma, X) = c(D, X)$.

Example 4.3 (i) Let $X = \mathbb{P}^n$ with D the class of a hyperplane. Then $K_X = -(n+1)D$, and so clearly $c(D,X) = 1/(n+1)$.

(ii) Let X be a primitive Fano variety, that is, a smooth projective variety with ample anticanonical class and $\operatorname{Pic} X = \mathbb{Z}K_X$. For D an effective divisor on X, we have that $D \in |-nK_X|$ for some $n \geq 1$, so that $c(D,X) = n$.

Definition 4.4 For a movable linear system Σ on a variety X, the ***virtual threshold of canonical adjunction*** is defined by the formula:

$$c_{\text{virt}}(\Sigma) = \inf_{\chi : X^\sharp \to X} \{c(\Sigma^\sharp, X^\sharp)\},$$

where the infimum is taken over all birational morphisms $\chi : X^\sharp \to X$, and $\Sigma^\sharp$ is the strict transform of Σ on $X^\sharp$ with respect to χ.

Clearly, the virtual threshold is a birational invariant of the pair (X, Σ): if $\chi : X \dashrightarrow X^+$ is a birational map, and $\Sigma^+ = \chi_* \Sigma$ is the strict transform of the linear system Σ with respect to χ^{-1}, then it follows that $c_{\text{virt}}(\Sigma) = c_{\text{virt}}(\Sigma^+)$.

Proposition 4.5 *Let X be a rationally connected variety, and D any divisor on X. Then the linear system $|D + \alpha K_X| = \emptyset$ for $\alpha \gg 0$. Furthermore, $\kappa(X) = -\infty$.*

Proof Since X is rationally connected, it follows that X is uniruled, and so there is a family of rational curves sweeping out X, that is to say, the set-theoretic union of the curves in this family contains a non-empty Zariski open (dense) subset of X. To see this, fix a general point x, and since for any general point y there is an irreducible rational curve joining x and y, it follows that there is a family with this property [8]. Let T be the variety parameterising these curves, and $Z \subset X \times T$ a cycle such that the generic fibre F of the restricted projection $\pi : Z \to T$ is a rational curve. Let D be a divisor on X. Then it follows from the adjunction formula that

$$K_X.F \leq \deg K_F = \deg K_{\mathbb{P}^1} = -2. \tag{22}$$

Now, suppose there exists a non-zero divisor $D' \subset X$ with $D' \in |D + \alpha K_X|$. Choose a rational curve F such that $F \not\subset D'$. Then

$$0 \leq D'.F = (D + \alpha K_X).F \leq D.F - 2\alpha.$$

Hence for $\alpha > (D.F)/2$, this is a contradiction. It therefore follows that the linear system $|D + \alpha K_X|$ is empty for $\alpha \gg 0$.

Now, for $m \geq 1$, suppose there exists a non-zero divisor $D'' \subset X$ with $D'' \in |mK_X|$. Then for a rational curve F, with $F \not\subset D''$, it follows from (22) that

$$0 \leq D''.F = (mK_X).F < 0,$$

a contradiction, and so $\kappa(X) = -\infty$. □

The above proposition means that as well as there being no non-zero differential forms on rationally connected varieties, they also satisfy the condition of *termination of adjunction*, with threshold as in definition 4.2.

Definition 4.6 (i) The variety X is said to be ***birationally superrigid*** if for any movable linear system Σ on X, the following equality holds:

$$c_{\text{virt}}(\Sigma) = c(\Sigma, X).$$

(ii) The variety X is said to be ***birationally rigid*** if for any movable linear system Σ on X, there exists a birational self-map $\chi \in \operatorname{Bir}(X)$ such that

$$c_{\text{virt}}(\Sigma) = c(\chi_* \Sigma, X).$$

Example 4.7 A smooth three-dimensional quartic $V = V_4 \subset \mathbb{P}^4$ is birationally superrigid - see [9].

4.2 Singularities of linear systems

We first recall the definition of a geometric discrete valuation $\nu : \mathbb{C}(X)^* \to \mathbb{Z}$, having a centre on X. Let $D \subset X$ be a prime divisor, $D \not\subset \operatorname{Sing} X$. Then D determines a discrete valuation $\nu_D = \operatorname{ord}_D$.

Definition 4.8 A ***realisation*** of a discrete valuation ν is a triple (X^+, φ, H), where $\varphi : X^+ \to X$ is a birational morphism, and $H \not\subset \operatorname{Sing} X^+$ a prime divisor such that $\nu = \nu_H$.

Definition 4.9 A discrete valuation is said to be ***geometric*** if it is realised by a prime Weil divisor on some model of the field $\mathbb{C}(X)$ of rational functions. We denote the set of geometric discrete valuations by $N(X)$.

Now, suppose that a variety X is not superrigid. Then there exists a movable linear system Σ, satisfying the inequality

$$c_{\text{virt}}(\Sigma) < c(\Sigma, X).$$

That is to say, there exists a birational morphism $\varphi : X^+ \to X$ such that $c(\Sigma^+, X^+) < c(\Sigma, X)$, where Σ^+ is the strict transform of the linear system Σ on X^+. Now, φ cannot be an isomorphism in codimension 1, that is to say, cannot be an isomorphism outside a closed subset of codimension ≥ 2, since then for D an effective divisor on X, with D^+ its strict transform, the thresholds $c(D, X)$ and $c(D^+, X^+)$ would coincide. Hence, there exist exceptional divisors $E \subset X^+$ of the morphism φ, each of which determine a geometric discrete valuation $\mathrm{ord}_E(\cdot)$ of the field $\mathbb{C}(X)$ of rational functions. This valuation is independent of the model X^+ in the following sense: if $\varphi^\sharp : X^\sharp \to X$ is another birational morphism, where the birational map $(\varphi^\sharp)^{-1} \circ \varphi : X^+ \dashrightarrow X^\sharp$ is an isomorphism at a generic point of the divisor E, so that $(\varphi^\sharp)^{-1} \circ \varphi(E) = E^\sharp \subset X^\sharp$ is an exceptional divisor of the morphism $\varphi^\sharp$, then $\mathrm{ord}_E = \mathrm{ord}_{E^\sharp}$. The irreducible subvariety $\varphi(E) \subset X$ is called the *centre* of the discrete valuation ord_E.

A divisor on the variety X is given by local equations, and so by applying the valuation ord_E, we obtain the *multiplicity* $\nu_E(D) \in \mathbb{Z}_+$ of an effective divisor D with respect to E, where $\nu_E(D) \geq 1$ if and only if $\varphi(E) \subset \mathrm{Supp}\, D$. In this notation, it follows that

$$\varphi^*(D) = D^+ + \sum_i \nu_{E_i}(D) E_i, \tag{23}$$

where the $E_i \subset X^+$ are the exceptional prime divisors of φ. For the canonical class,

$$K_{X^+} = \varphi^*(K_X) + \sum_i a(E_i) E_i, \tag{24}$$

where $a(E_i) = a(E_i, X)$ is the *discrepancy*. Since X is terminal, $a(E_i, X) \geq 1$.

Definition 4.10 A geometric discrete valuation ord_E of the field $\mathbb{C}(X)$ is called a ***maximal singularity*** of the linear system Σ if the Noether-Fano inequality

$$\nu_E(\Sigma) > n a(E)$$

holds, where $\nu_E(\Sigma) = \nu_E(D)$ for a generic divisor $D \in \Sigma$, and $n = c(\Sigma, X)$. Note that we sometimes say that the exceptional divisor E is the maximal singularity, and take it to mean the same thing.

Note that it is enough for X^+ to be nonsingular at a generic point of the exceptional divisor E for the numbers $\nu_E(D)$ and $a(E)$ to be well defined: since the local ring of a nonsingular point is a UFD, there exists an open set $U \subset X^+$ intersecting E, consisting of nonsingular points, such that E is defined in U by a local equation $\pi \in k[U]$, and there exists an integer $k \geq 0$ with $f \in (\pi^k)$ and $f \notin (\pi^{k+1})$, where $f \in k[U]$ is the equation of $\varphi^*(D)$ in this set. This condition on X^+ is satisfied if, for example, X^+ is normal, since then it will be nonsingular in codimension 1.

Definition 4.11 An irreducible subvariety $Y \subset X$ of codimension ≥ 2 is called a ***maximal subvariety*** of the linear system Σ if the inequality

$$\mathrm{mult}_Y(\Sigma) > n(\mathrm{codim}\, Y - 1)$$

holds, where $\mathrm{mult}_Y(\Sigma) = \mathrm{mult}_Y(D)$ for a generic divisor $D \in \Sigma$.

To show that this definition is consistent with that of a maximal singularity, suppose that $Y \subset X$ is a maximal subvariety of the linear system Σ, and consider the blowup $\varphi : X^+ \to X$ at Y, with $E = \varphi^{-1}(Y)$ the exceptional divisor. Since φ is just a single blowup, it follows that $\nu_E(\Sigma) = \mathrm{mult}_Y(\Sigma)$, and $a(E) = \mathrm{codim}\, Y - 1$ follows from proposition 2.9. Hence, E already realises a maximal singularity of the linear system Σ, and so these are the simplest type of maximal singularities.

Definition 4.12 A maximal singularity is said to be ***infinitely near*** if it is not a maximal subvariety.

Theorem 4.13 *Let Σ be a movable linear system on X, satisfying the inequality $c_{\mathrm{virt}}(\Sigma) < c(\Sigma)$. Then Σ has a maximal singularity.*

Proof As above, let $\varphi : X^+ \to X$ be a birational morphism, with $c(\Sigma^+, X^+) < c(\Sigma, X) = n$ for Σ^+ the strict transform of the system on X^+. Let $D \in \Sigma$ be a generic divisor, with $D^+ \in \Sigma^+$ its strict transform on X^+. Then, from (23) and (24), we have that

$$\begin{aligned} D^+ + nK_{X^+} &= (\varphi^*(D) - \sum_i \nu_{E_i}(D)E_i) + n(\varphi^*(K_X) + \sum_i a(E_i)E_i) \\ &= \varphi^*(D + nK_X) - \sum_i (\nu_{E_i}(D) - na(E_i))E_i, \end{aligned}$$

where the $E_i \subset X^+$ are the exceptional prime divisors of φ. Since $c(\Sigma^+, X^+) < n$, $|D^+ + nK_{X^+}| = \emptyset$, and thus $D^+ + nK_{X^+}$ is not effective on X^+. By definition of $c(\Sigma, X)$, $D + nK_X$ is effective on X, and since the pullback of an effective divisor is effective, it follows that $-\sum_i (\nu_{E_i}(D) - na(E_i))E_i$ is not effective, because otherwise

$$\varphi^*(D + nK_X) - \sum_i (\nu_{E_i}(D) - na(E_i))E_i > 0,$$

which it isn't. If the $\nu_{E_i}(D) - na(E_i)$ are all negative, then it is effective, so one of them must be positive; that is to say, there exists an exceptional divisor E_j with $\nu_{E_j}(D) > na(E_j)$, as required. $\square$

Let $E^\sharp$ be an exceptional divisor of the birational map $\psi : X^\sharp \dashrightarrow X$, satisfying the Noether-Fano inequality, $\nu_{E^\sharp}(\Sigma) > na(E^\sharp)$, such that ψ contracts $E^\sharp$ to a subvariety $B = \psi(E^\sharp) \subset X$ of codimension ≥ 2, where $B \not\subset \operatorname{Sing} X$ is not strictly contained in the singular locus of X. Let $\varphi : X_1 \to X$ be the blowup of the subvariety B, with exceptional divisor $E = \varphi^{-1}(B)$.

Proposition 4.14 *Either the composition $\varphi^{-1} \circ \psi : X^\sharp \dashrightarrow X_1$ is an isomorphism in a neighbourhood of a generic point of $E^\sharp$, so that $\varphi^{-1} \circ \psi(E^\sharp) = E$ and so $\nu_{E^\sharp} = \nu_E$, or $\varphi^{-1} \circ \psi(E^\sharp) = B_1$, where B_1 is an irreducible subvariety of* codimension ≥ 2, *$B_1 \not\subset \operatorname{Sing} X_1$, $B_1 \subset E$, and $\varphi(B_1) = B$.*

Proof These are the only two possibilities. Note that $\varphi(B_1) = \varphi(\varphi^{-1} \circ \psi(E^\sharp)) = \psi(E^\sharp) = B$, so that B_1 does not just consist of fibres of φ in E. □

We can therefore use this construction to obtain a sequence of blowups

$$\begin{array}{cccc} \varphi_{i,i-1}: & X_i & \longrightarrow & X_{i-1} \\ & \cup & & \cup \\ & E_i & \longrightarrow & B_{i-1} \end{array}$$

where $i = 1, 2, \ldots$, $B = B_0$, $X = X_0$, $E = E_1$, B_j is the centre of $E^\sharp$ on X_j, and B_{i-1} is the centre of the blowup $\varphi_{i,i-1}$, with exceptional divisor $E_i = \varphi_{i,i-1}^{-1}(B_{i-1})$. For $i > j$, let

$$\varphi_{i,j} = \varphi_{j+1,j} \circ \ldots \circ \varphi_{i,i-1},$$

with $\varphi_{i,i}$ the identity map on X_i. It follows from the above proposition that $\varphi_{i,j}(B_i) = B_j$ for $i > j$.

Proposition 4.15 *The sequence of blowups detailed above terminates; that is to say, for some $K \geq 1$, $\varphi_{K,0}^{-1} \circ \psi(E) = E_K$.*

Proof The decomposition of the discrepancies $a(E_i, X)$ of the exceptional divisors E_i, with respect to the model X, described in proposition 5.6(ii) shows that they are strictly increasing; in particular, that $a(E_i, X) \geq i$. On the other hand, $\varphi_{i,0}^{-1} \circ \psi(E^\sharp) = B_i \subset E_i \subset X_i$, and so the centre of $E^\sharp$ on X_i is contained in E_i. It follows that $a(E_i, X) \leq a(E^\sharp, X)$: consider the composition of maps

$$X^\sharp \xrightarrow[\varphi_{i,0}^{-1} \circ \psi]{} X_i \xrightarrow[\varphi_{i,0}]{} X.$$

Then

$$K_{X_i} = \varphi_{i,0}^*(K_X) + \sum_{k=1}^{i} a(E_k^i, X) E_k^i,$$

so that

$$\begin{aligned} K_{X^\sharp} &= (\varphi_{i,0}^{-1} \circ \psi)^*(K_{X_i}) + \sum_j a(E'_j, X_i)E'_j \\ &= \psi^*(K_X) + \sum_{k=1}^{i} a(E^i_k, X)(\varphi_{i,0}^{-1} \circ \psi)^*(E^i_k) + \sum_j a(E'_j, X_i)E'_j, \end{aligned}$$

where the $E_k \subset X_k$ and $E'_j \subset X^\sharp$ are the exceptional divisors of $\varphi_{i,0}$ and $\varphi_{i,0}^{-1} \circ \psi$, respectively. Hence,

$$a(E^\sharp, X) \geq a(E^\sharp, X_i) + a(E_i, X)\,\mathrm{ord}_{E^\sharp}(\varphi_{i,0}^{-1} \circ \psi)^*(E_i) \geq a(E_i, X),$$

where the first inequality accounts for the fact that the centre of $E^\sharp$ on X_i may also be contained in some $E_l \neq E_i$.

It then follows from the strictly increasing nature of the discrepancies $a(E_i, X)$ that there must exist an integer K with $a(E_K, X) = a(E^\sharp, X)$, and so the sequence of blowups terminates. □

Definition 4.16 The sequence of blowups just described is called the ***resolution of the discrete valuation*** $\nu_{E^\sharp}$ with respect to the model X.

4.3 The oriented graph structure

On the set of exceptional divisors $\{E_1, \ldots, E_K\}$, we introduce an oriented graph structure as follows. Let $i \to j$ if $i > j$ and $B_{i-1} \subset E_j^{i-1}$, where E_j^{i-1} denotes the strict transform of E_j on X_{i-1}. Let $p_{i,j}$ denote the number of paths from E_i to E_j for $i > j$ in the oriented graph just described, and set $p_{i,i} = 1$. Let $\nu_j = \mathrm{ord}_{E_j} \varphi_{j,j-1}^*(\Sigma^{j-1})$, so $\mathrm{ord}_{E_j} \varphi_{j,j-1}^*(\Sigma^{j-1}) = \mathrm{mult}_{B_{j-1}} \Sigma^{j-1}$ if the centre of the blowup, B_{j-1}, is smooth.

4.4 The Noether-Fano inequality reformulated

In chapter 6, we also use the Noether-Fano inequality formulated in the language of $\mathbb{Q}$-divisors, and so the terminology and notions required for this are explained below.

Definition 4.17 A $\mathbb{Q}$***-divisor*** D on X is a finite formal linear combination

$$D = \sum_i d_i D_i$$

of codimension 1 irreducible subvarieties $D_i \subset X$ with rational coefficients $d_i \in \mathbb{Q}$.

A $\mathbb{Q}$-divisor D is called integral if the coefficients d_i are integers. If D is a $\mathbb{Q}$-divisor, then an integer m is said to clear the denominators of D if mD is integral.

Definition 4.18 A $\mathbb{Q}$-divisor D is ***$\mathbb{Q}$-Cartier*** if for some $m \in \mathbb{Z}$, mD is an (integral) Cartier divisor. It follows that in a factorial variety, any $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier.

As an application of these definitions, suppose that $f : Y \to X$ is a morphism of irreducible varieties, and D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, with $f(Y) \not\subset \operatorname{Supp} D$. Then the pullback $f^*(D)$ is determined as a $\mathbb{Q}$-divisor by clearing denominators.

Definition 4.19 A ***pair*** (X, D) consists of a variety X and an effective $\mathbb{Q}$-divisor D on X.

Definition 4.20 Let $D \in \Sigma \subset X$ be a generic divisor. Then for $n = c(\Sigma, X)$, the pair $(X, \frac{1}{n}D)$ is ***not canonical*** if there exists a birational morphism $\psi : X^\sharp \to X$, and an exceptional divisor $E^\sharp \subset X^\sharp$, such that the following inequality

$$\nu_{E^\sharp}\left(\frac{1}{n}D\right) > a(E^\sharp)$$

holds. The exceptional divisor $E^\sharp$ is called a ***non-canonical singularity*** of the pair $(X, \frac{1}{n}D)$. The pair $(X, \frac{1}{n}D)$ is ***not log canonical*** if the inequality

$$\nu_{E^\sharp} > a(E^\sharp) + 1$$

holds, and $E^\sharp$ is said to be a ***non-log canonical singularity*** of the pair.

Now that we have defined a pair (X, D), we give two more definitions which will be required in chapter 6.

Definition 4.21 Let X be a smooth variety. A reduced effective divisor $D = \sum_i D_i$ on X is said to be a ***normal crossing divisor*** if for each closed point $x \in X$, a local equation of D at x can be written as a product $f = u_1 \dots u_r$ in $\mathcal{O}_x$, where functions $u_1, \dots, u_r \in \mathcal{O}_x$ form part of a regular system of local parameters at x. A $\mathbb{Q}$-divisor $\sum_i d_i D_i$ has ***normal crossing support*** if $\sum_i D_i$ is a normal crossing divisor.

Definition 4.22 A ***resolution of singularities*** of the pair (X, D) is a projective birational morphism $f : \widetilde{X} \to X$ with $\widetilde{X}$ nonsingular, such that the $\mathbb{Q}$-divisor $f^*(D) + \sum_i E_i$ has normal crossing support, where the E_i are the exceptional divisors of f.

Remark 4.23 At the beginning of section 4.1, rather than imposing the condition that X was smooth, we instead supposed that X was a $\mathbb{Q}$-factorial Fano variety, with at worst terminal singularities. Being $\mathbb{Q}$-factorial means that every Weil divisor D on X is $\mathbb{Q}$-Cartier, that is to say, for some $m \in \mathbb{N}$, mD is Cartier, so that mD is given locally by a single equation enabling us to pull back such divisors. Having only terminal singularities means that the canonical class, K_X, is $\mathbb{Q}$-Cartier, and that if $f : \widetilde{X} \to X$ is a resolution of singularities of X, then the discrepancies of the exceptional prime divisors E_i in the ramification formula $K_{\widetilde{X}} = f^*(K_X) + \sum_i a(E_i)E_i$, are strictly positive. Hence, the definitions in this chapter make sense under this condition on X.

4.5 The method of maximal singularities

The threshold of canonical adjunction of a linear system Σ on X is fundamental in the method of maximal singularities. If for any linear system Σ on X, $c_{\text{virt}}(\Sigma) = c(\Sigma, X)$, then by definition X is birationally superrigid. Also, we saw in theorem 4.13 that if there exists a linear system Σ on X satisfying the inequality $c_{\text{virt}}(\Sigma) < c(\Sigma, X)$, then Σ has a maximal singularity. Hence, the next step in the method is to determine which maximal singularities can be realised on the variety X and which cannot. That is to say, for each geometric discrete valuation $\nu = \nu_E \in N(X)$, realised by an exceptional divisor $E \subset X^+$ on some model $\varphi : X^+ \to X$, we determine for each movable linear system Σ on X, with threshold $c(\Sigma)$, whether ν_E is a maximal singularity, satisfying the Noether-Fano inequality $\nu_E(\Sigma) > c(\Sigma)a(E)$. By showing that this inequality is not satisfied, we *exclude* the given maximal singularity.

Now, if a maximal singularity cannot be excluded, then we explicitly describe a movable linear system Σ with the maximal singularity E. The next step is then to try to *untwist* this maximal singularity via birational maps that simplify the linear system Σ. The simple cases are when the untwisting birational maps are birational automorphisms. If untwisting is possible, the procedure in the simple cases is as follows: for a linear system Σ with maximal singularity E, we construct a birational automorphism $\tau_E \in \mathrm{Bir}(X)$ such that

$$c((\tau_E^{-1})_*\Sigma) < c(\Sigma),$$

where $(\tau_E^{-1})_*\Sigma$ is the strict transform of Σ with respect to τ_E, such that E is no longer a maximal singularity of the linear system $(\tau_E^{-1})_*\Sigma$. Hence, τ removes the maximal singularity E whilst decreasing the threshold of canonical adjunction.

Now, consider a primitive Fano variety X, and assume that the untwisting procedure is successful for any maximal singularities not excluded in the previous step. Note that if all maximal singularities can be excluded, then it follows from theorem 4.13 that the variety is birationally superrigid. Therefore, suppose that there is at least one maximal singularity which requires untwisting.

Theorem 4.24 (i) *X is birationally rigid;*

(ii) *The group $\mathrm{Bir}(X)$ of birational automorphisms of X is generated by the subgroup $\mathrm{Aut}(X)$ of biregular automorphisms and the subgroup $B(X)$ generated by the untwisting maps τ_E corresponding to the maximal singularities E.*

Proof (i) Since X is a primitive Fano variety, with $\operatorname{Pic} X = \mathbb{Z}K_X$, it follows that for $\Sigma \subset |-nK_X|$ a linear system on X, $c(\Sigma, X) = n$, so that the threshold of an arbitrary linear system on X is a positive integer. Hence, the untwisting procedure is finite, so that if $E_1, \dots, E_m$ are a sequence of maximal singularities of a linear system Σ, we can construct a birational automorphism $\tau = \tau_{E_1} \circ \cdots \circ \tau_{E_m}$ such that $c((\tau^{-1})_*\Sigma) < c(\Sigma)$, and so that the system $\Sigma^\tau = (\tau^{-1})_*\Sigma$ has no maximal singularities. It then follows from theorem 4.13 that $c_{\text{virt}}(\Sigma) = c_{\text{virt}}(\Sigma^\tau) = c(\Sigma^\tau)$. Iterating this procedure for each linear system with maximal singularities, it follows that X is birationally rigid.

(ii) Consider a linear system $\Sigma = |-nK_X|$, and its strict transform $\Sigma^\chi = (\chi^{-1})_*\Sigma \subset |-mK_X|$ for an arbitrary birational automorphism $\chi \in \mathrm{Bir}(X)$. First, suppose $c(\Sigma^\chi) = c(\Sigma)$, so that $m = n$. Since χ is birational, $\dim \Sigma^\chi = \dim \Sigma$, but Σ is the complete linear system $|-nK_X|$, so that $\Sigma^\chi = \Sigma$, and so $\chi \in \mathrm{Aut}(X)$. Next, suppose that $c_{\text{virt}}(\Sigma^\chi) \le c(\Sigma^\chi) < c(\Sigma)$. Either $c_{\text{virt}}(\Sigma^\chi) = c(\Sigma^\chi)$, so that χ is already an untwisting map, else $c_{\text{virt}}(\Sigma^\chi) < c(\Sigma^\chi)$, and so it follows from theorem 4.13 that Σ^χ has a maximal singularity. Hence, by assumption there exists an untwisting map τ so that the system $(\tau^{-1})_*\Sigma^\chi = \Sigma^{\chi\circ\tau}$ has no maximal singularities, and it follows from theorem 4.13 that

$$c_{\text{virt}}(\Sigma^{\chi\circ\tau}) = c(\Sigma^{\chi\circ\tau}) < c(\Sigma^\chi) < c(\Sigma).$$

Hence, $\chi\circ\tau$ is an untwisting map, and so $\chi \in B(X)$. Finally, suppose $c(\Sigma^\chi) > c(\Sigma)$. It then follows from

$$c_{\text{virt}}(\Sigma^\chi) \le c(\Sigma^\chi) > c(\Sigma) \ge c_{\text{virt}}(\Sigma) = c_{\text{virt}}(\Sigma^\chi)$$

that $c_{\text{virt}}(\Sigma^\chi) < c(\Sigma^\chi)$ so that Σ^χ has a maximal singularity, and we are therefore back in the previous case. $\square$

Remark 4.25 If it cannot be proved that a potentially realisable maximal singularity can be excluded or untwisted, we cannot say anything significant about the birational geometry of the variety.

4.6 Excluding maximal singularities

We carry forward the notation of section 4.2. Let X be a smooth variety, and $E^\sharp$ an exceptional divisor of the birational map $\psi : X^\sharp \dashrightarrow X$, so that $E^\sharp$ is a maximal singularity of a linear system $\Sigma \subset |-nK_X|$ on X, and ψ contracts $E^\sharp$ to a subvariety $B \subset X$ of codimension ≥ 2. Furthermore, let the resolution of the discrete valuation $\nu_{E^\sharp}$ be given by the sequence of blowups $\varphi_{i,i-1} : X_i \to X_{i-1}$, with B_{i-1} the centre of the blowup $\varphi_{i,i-1}$.

By construction of the resolution, the dimensions of successive centres are non-decreasing, so that $\dim B_0 = \ldots = \dim B_{K-1}$ or $\dim B_0 < \dim B_{K-1}$. From proposition 2.9, the discrepancy of the blowup $\varphi_{i,i-1} : X_i \to X_{i-1}$, with smooth centre B_{i-1}, is $\delta_i = \operatorname{codim} B_{i-1} - 1$. Hence, if $\dim B_0 = \dim B_{K-1}$, then $\delta_1 = \ldots = \delta_K = \delta = \operatorname{codim} B_0 - 1$. Set $\nu_j = \operatorname{ord}_{E_j} \varphi_{j,j-1}^*(\Sigma^{j-1}) = \operatorname{mult}_{B_{j-1}} \Sigma^{j-1}$. We shall show later, in proposition 5.6, that the Noether-Fano inequality $\nu_{E^\sharp}(\Sigma) > na(E^\sharp)$ can be rewritten as

$$\sum_{j=1}^{K} p_{K,j}(\nu_j - n\delta) > 0. \tag{25}$$

Since the multiplicities $\nu_1 \geq \ldots \geq \nu_K$ are non-increasing (the maps $\varphi_{i,i-1}$ are blowups), it follows from (25) that

$$\nu_1 > n\delta, \tag{26}$$

so that the subvariety B_0 is a maximal subvariety of Σ. We now give an example to illustrate the exclusion of a maximal subvariety of codimension 2.

Example 4.26 Let $\pi : V \to \mathbb{P}^m \supset W_{2m}$ for $m \geq 3$ be a smooth double space of index 1 branched over a smooth hypersurface W of degree $2m$, so that $\operatorname{Pic} V = \mathbb{Z}K_V$ where $K_V = -\pi^* H$ for H the class of a hyperplane in $\mathbb{P}^m$. Let B_0 be an irreducible subvariety of codimension 2 in V. Then B_0 cannot be a maximal subvariety of a movable linear system $\Sigma \subset |-nK_V|$.

Suppose that Σ has a maximal subvariety B_0 of codimension 2. Let $D_1, D_2 \in \Sigma$ be generic divisors, so that $\operatorname{codim}(D_1 \cap D_2) = 2$, with no common components since the system is moving. Let $Z = D_1 \circ D_2$ be the effective cycle of the (scheme-theoretic) intersection of D_1 and D_2. Hence,

$$Z = aB_0 + Y,$$

where Y is an effective cycle of codimension 2, and $a > n^2$ by (26). By the Lefschetz theorem, the Chow group A^2V of cycles of order 2 is $\mathbb{Z}K_V^2$, so that $B_0 \sim rK_V^2$, for some $r \geq 1$, $Y \sim sK_V^2$, for some $s \geq 0$ and $Z \sim n^2K_V^2$. Hence, the anticanonical degree of Z is

$$\deg Z = (Z \cdot (-K_V)^{m-2}) = \left(n^2(\pi^*H)^2 \cdot (\pi^*H)^{m-2}\right) = 2n^2,$$

where $(\pi^*H)^m = 2$ since V is a double cover. But $Z = aB_0 + Y \sim arK_V^2 + sK_V^2$, so that

$$2n^2 = \left(ar(\pi^*H)^2 \cdot (\pi^*H)^{m-2} + s(\pi^*H)^2 \cdot (\pi^*H)^{m-2}\right) = 2ar + 2s.$$

Hence, $n^2 = ar + s > rn^2 + s$, a contradiction.

Note that if B_0 is a maximal subvariety of codimension ≥ 3, it follows from (26) that $\operatorname{mult}_{B_0} \Sigma > 2n$, whereas the anticanonical degree of Σ is $\deg \Sigma = (\Sigma \cdot (-K_V)^{m-1}) = 2n$, a contradiction.

5 Main theorem: a particular case

Now we have all of the tools necessary to state and prove the main result of this thesis; informally, that under certain conditions, the double space of index 1 branched over a generic hypersurface in the projective space containing a singular locus of high dimension, is birationally superrigid. In this chapter, we deal with a particular case which enables us to proceed in a similar way to that of when the branch divisor is smooth. We also state and prove the standard corollaries of birational superrigidity of the variety.

5.1 Formulation of the main result

The integer $m \geq 6$ is fixed throughout. Fix a linear subspace $P \subset \mathbb{P}^m$ of codimension k satisfying the inequality $m \leq \frac{k(k-3)}{2} + 2$. Let $W = W_{2m} \subset \mathbb{P}^m$ be a generic hypersurface singular at every point of P (so that $P = \operatorname{Sing} W$ is the locus of double points of W). Explicitly, if we choose equations $x_0 = \ldots = x_{k-1} = 0$ for P, and coordinates $x_k, \ldots, x_m$ on the plane, then W is given by the equation

$$f(x_*) = \sum_{i,j=0}^{k-1} x_i x_j h_{ij}(x_*) = 0,$$

where $h_{ij}(x_*) = h_{ji}(x_*)$ are homogeneous polynomials of degree $2m-2$ in $x_0, \ldots, x_m$. Let $\pi : V \to \mathbb{P}^m \supset W_{2m}$ be the double space of index 1 branched over W, which can be defined explicitly as the hypersurface given by the equation

$$x_{m+1}^2 = f(x_0, \ldots, x_m)$$

in the weighted projective space $\mathbb{P}^{m+1}(1, \ldots, 1, m)$.

Proposition 5.1 *The variety V is a factorial m-dimensional variety, the singular locus of which is $P^* = \pi^{-1}(P)$. Moreover, V is a Fano variety of index one, that is, $\operatorname{Pic} V = \mathbb{Z}K_V$ where $K_V = -\pi^* H$, for H the class of a hyperplane in $\mathbb{P}^m$.*

Proof These properties of V follow from the fact that the double cover $\pi : V \to \mathbb{P}^m$, branched over W, extends to a double cover $\pi : V^+ \to \widetilde{\mathbb{P}^m}$, branched over $\widetilde{W}$, where $\varphi : V^+ \to V$ and $\widetilde{\varphi} : \widetilde{\mathbb{P}^m} \to \mathbb{P}^m$ are the blowups of P^* and P, respectively, and $\widetilde{W}$ is the strict transform of W by $\widetilde{\varphi}$.

Explicitly, recall that $P \subset \mathbb{P}^m$ is of codimension k, so that $P^* \subset V$ has codimension k. For ease of notation, we will denote the exceptional divisor of both blowups φ and $\widetilde{\varphi}$ by E^+. Also, for H the class of a hyperplane in $\mathbb{P}^m$, denote its pullback onto $\widetilde{\mathbb{P}^m}$ by H. We prove existence of the commutative diagram

$$\begin{array}{ccc} E^+ & & \\ \cap & & \\ V^+ & \xrightarrow{\varphi} & V \\ \pi\downarrow & & \downarrow\pi \\ \widetilde{W} \subset \widetilde{\mathbb{P}^m} & \xrightarrow[\widetilde{\varphi}]{} & \mathbb{P}^m \supset W \end{array}$$

Choose a local system of parameters $z_0, \ldots, z_{m-1}$ at a generic point $p \in P$, so that P is defined by equations $z_0 = \ldots = z_{k-1} = 0$, and W is defined locally by

$$f(z_0, \ldots, z_{m-1}, 1) = \sum_{i,j=0}^{k-1} z_i z_j h_{ij}(z_*) = 0,$$

where $h_{ij}(z_*) = h_{ji}(z_*)$ are polynomials of degree $\leq 2m-2$ in $z_0, \ldots, z_{m-1}$. Then V can be defined locally at $v = \pi^{-1}(p)$ by the equation

$$z_{m+1}^2 = f(z_0, \ldots, z_{m-1}, 1),$$

where $z_0, \ldots, z_{m-1}, z_{m+1}$ is the system of local parameters of $\mathbb{P}^m \times \mathbb{A}^1$. By blowing up P at the bottom and P^* at the top, we obtain

$$u_0^2 \left(u_{m+1}^2 - \sum_{i,j=0}^{k-1} u_i u_j h_{ij}(u_*) \right) = 0$$

for the equation of the total transform of V, with the equation of the strict transform

$$u_{m+1}^2 = \sum_{i,j=0}^{k-1} u_i u_j h_{ij}(u_*)$$

defining the double space V^+ in the standard chart $U_0 = (z_0 \neq 0)$, where $z_0 = u_0$, $z_i = u_i u_0$ for $i = 1, \ldots, k-1$, $z_j = u_j$ for $j = k, \ldots, m-1$, and $z_{m+1} = u_{m+1} u_0$ are the equations of the blowup, and $h_{ij}(u_*) = h_{ij}(u_0, u_1 u_0, \ldots, u_{k-1} u_0, u_k, \ldots, u_{m-1}, 1)$.

The double space V^+ is smooth: consider the linear system of branch divisors $W \subset \mathbb{P}^m$ containing P, and the intersection of their strict transforms on $\widetilde{\mathbb{P}^m}$ with the resulting exceptional divisor. That is to say, consider the linear system of divisors on E^+ of the form $\widetilde{W_h} \cap E^+$, each defined in U_0 by an equation of the form

$$\sum_{i,j=0}^{k-1} u_i u_j h_{ij}(\overline{u}_*) = 0,$$

for $h_{ij}(\overline{u}_*) = h_{ij}(0, \ldots, 0, u_k, \ldots, u_{m-1}, 1)$. Clearly, this linear system is base point free, and so by Bertini's Theorem, a generic element of this linear system is smooth.

It follows from the genericity of W that $\widetilde{W} \cap E^+$ is smooth, and so $\widetilde{W} \subset \widetilde{\mathbb{P}^m}$ is smooth. Hence, V^+ is smooth.

Now, we have that $K_{\mathbb{P}^m} = -(m+1)H$. It therefore follows from proposition 2.9 that

$$K_{\widetilde{\mathbb{P}^m}} = -(m+1)H + (k-1)E^+.$$

We have that $\operatorname{Pic} \widetilde{\mathbb{P}^m} = \mathbb{Z}H \oplus \mathbb{Z}E^+$, and $W \sim 2mH$ as W is a hypersurface of degree $2m$. Since $P = \operatorname{Sing} W$ is the locus of double points of W, we have that

$$\begin{aligned} \widetilde{W} &= \widetilde{\varphi}^*(W) - 2E^+ \\ &\sim 2mH - 2E^+. \end{aligned}$$

Consider the strict transform D^+ on V^+ of a Weil divisor D on V. Since V^+ is smooth, it follows that V^+ is factorial, and by the Lefschetz theorem, $\operatorname{Pic} V^+ = \mathbb{Z}\pi^*(H) + \mathbb{Z}E^+$. Hence D^+ is Cartier, and it follows that V is factorial. Now, due to the double cover, we get

$$\begin{aligned} K_{V^+} &= \pi^*(K_{\widetilde{\mathbb{P}^m}}) + \frac{1}{2}\pi^*(\widetilde{W}) \\ &= -(m+1)\pi^*(H) + (k-1)\pi^*(E^+) + \frac{1}{2}(2m\pi^*(H) - 2\pi^*(E^+)) \\ &= -\pi^*(H) + (k-2)E^+, \end{aligned}$$

and so $K_V = -\pi^*(H)$, as required. □

Now, the main result is as follows.

Theorem 5.2 *V is birationally superrigid.*

The theory of birational rigidity goes back to the classical paper [9] of Iskovskikh and Manin on three-dimensional quartics. The basic definitions of theory are detailed in chapter 4; see [31] also. The theorem above means that for any movable linear system Σ on V, its virtual threshold of canonical adjunction $c_{virt}(\Sigma, V)$ is equal to the threshold $c(\Sigma, V)$ on V. Given this equality, the claims of Corollary 5.3 follow in the standard way [31].

Corollary 5.3 (i) *V cannot be fibred into uniruled varieties by a non-trivial rational map.*

(ii) *V is non-rational.*

(iii) *The groups of birational and biregular self-maps coincide:* $\operatorname{Bir} V = \operatorname{Aut} V = \mathbb{Z}/2\mathbb{Z}$.

Proof of Corollary 5.3 (i) If

$$\begin{array}{ccc} \chi : V & \longrightarrow & V' \\ & & \downarrow{\scriptstyle \pi} \\ & & S' \end{array}$$

is a birational map onto V', where $\dim S' \geq 1$, and fibres of π are uniruled, then we choose Σ' to be the pullback of a moving linear system on S'. That is to say, let D' be an effective Cartier divisor on S', and let $\Sigma' = \pi^*(D')$. Define $\Sigma = (\chi^{-1})_*\Sigma'$ to be the strict transform of Σ' on V with respect to χ. Since V is Fano, with $\operatorname{Pic} V = \mathbb{Z}K_V$, it follows that $c(\Sigma, V) > 0$. We claim that $c(\Sigma', V') = 0$: suppose there are $a, b > 0$ such that $a\pi^*(D') + bK_{V'}$ is effective on V'. Let F be a general fibre of π, so that F is uniruled and not contained in the divisor $a\pi^*(D') + bK_{V'}$. Then, by the adjunction formula,

$$(a\pi^*(D') + bK_{V'})\,|_F = (bK_{V'})|_F = bK_F,$$

so that K_F is effective, a contradiction since F is uniruled. Hence, $c_{\text{virt}}(\Sigma, V) = c(\Sigma', V') = 0$, and so by superrigidity, we have that $c(\Sigma, V) = 0$. (ii) This follows immediately from (i). (iii) This follows from theorem 4.24 (ii) and [16]. □

The following existing results are closely related to our work.

Example 5.4 (i) For $m \geq 3$, a smooth double space $\pi : V \to \mathbb{P}^m \supset W_{2m}$ of index 1 branched over a smooth hypersurface W of degree $2m$ is birationally superrigid - see [32].

(ii) The setup is as in (i), but W has a unique singularity at the point $x \in W$, with $\operatorname{mult}_x W = 2l$, $l \leq m-2$, and the tangent cone T_xW defines a smooth hypersurface of degree $2l$ in the projective space $\mathbb{P}(T_x\mathbb{P}^m) \cong \mathbb{P}^{m-1}$. Then V is birationally superrigid - see [33]. (For the case $m = 3$, W is chosen to be *maximally generic* - see [33, proposition 1] for details.)

(iii) The setup is as in (i), but W has multiple isolated singularities, each satisfying the conditions in (ii). Then V is birationally superrigid - see [34].

(iv) Birational geometry of 3-fold double spaces with a double line is more complicated as they can be fibred into del Pezzo surfaces - see [35].

5.2 Plan and start of the proof

Assume that V is not superrigid. By definition, this means that we can fix a movable linear system $\Sigma \subset |-nK_V|$ such that $c_{virt}(\Sigma) < c(\Sigma, V) = n$. It follows from theorem 4.13 that the linear system Σ has a maximal singularity. That is to say, for some irreducible exceptional divisor $E^\sharp \subset V^\sharp$ on some model $\psi : V^\sharp \to V$ of V, the Noether-Fano inequality $\operatorname{ord}_{E^\sharp} \psi^*(\Sigma) > na(E^\sharp)$ holds. In other words, the pair $(V, \frac{1}{n}\Sigma)$ is not canonical, and $E^\sharp$ is a non-canonical singularity of that pair. Let $B = \psi(E^\sharp)$ be its centre on V. Now, arguments identical to those in [31] exclude the possibility of $B \not\subset P^* = \operatorname{Sing} V$: firstly, if $\operatorname{codim} B = 2$, we may repeat word for word the proof given in example 4.26 (alternatively, see[31, chapter 2, proposition 2.9]) since $\operatorname{codim} P^* \geq 5$; that is to say, a generic point of B lies outside the singular locus. Otherwise, $\operatorname{codim} B \geq 3$, and calculating at the generic point of B, which is nonsingular on V, we get the inequality

$$\operatorname{mult}_B Z > 4n^2 \tag{27}$$

for the self-intersection $Z = (D_1 \circ D_2)$ of the linear system Σ - see [31, chapter 2, theorem 2.2] for details. The inequality (27) is impossible since $\deg Z = 2n^2$. Therefore, we may assume that $B \subset \operatorname{Sing} V = P^*$.

5.3 Maximal singularities over a subvariety in the singular locus

As we mentioned above, it follows from the genericity of W that the blowup $\varphi : V^+ \to V$ of the singular locus P^* is a smooth variety, with the exceptional divisor $E^+ = \varphi^{-1}(P^*)$. The subvariety E^+ is smooth and $\varphi_{E^+} : E^+ \to P^* \cong P$ is a fibration into quadrics. Writing, in the notation above, $h_{ij}(x_0, \dots, x_m) = P_{ij}(x_k, \dots, x_m) + h_{ij}^+$, where each term of the homogeneous polynomials h_{ij}^+ contains at least one of the variables $x_0, \dots, x_{k-1}$, we can identify the fibre of φ_{E^+} over a point $(0 : \dots : 0 : a_k : \dots : a_m)$ with the quadric hypersurface

$$x_{m+1}^2 = \sum_{i,j=0}^{k-1} x_i x_j P_{ij}(a_k, \dots, a_m).$$

As a special case, let us consider first the case when $B = P^*$ is the whole singular locus. In our proof of the general case in the following chapter, the main technical problem comes from the fact that some fibres of E^+/P^* can be singular. We will therefore need to determine precisely how degenerate the quadric fibres can be in order to complete the proof in this case.

5.4 The case when the centre is the whole singular locus

To exclude this case, we need arguments which are similar to those of the smooth case. We carry forward the notation, and modify the computations in [31] as follows: the resolution with respect to the model $V = V_0$ is a sequence of blowups

$$\begin{array}{cccc} \varphi_{i,i-1}: & V_i & \longrightarrow & V_{i-1} \\ & \cup & & \cup \\ & E_i & \longrightarrow & B_{i-1} \end{array}$$

where $i = 1, 2, \ldots, K$, $B = B_0$, $V^+ = V_1$, $E^+ = E_1$, B_j is the centre of $E^\sharp$ on V_j, B_{i-1} is the centre of the blowup $\varphi_{i,i-1}$, with exceptional divisor $E_i = \varphi_{i,i-1}^{-1}(B_{i-1})$, and we have that $\varphi_{K,0}^{-1} \circ \psi(E^\sharp) = E_K$. We divide the resolution $\varphi_{i,i-1}: V_i \to V_{i-1}$ into two parts: $i = 1, \ldots, L \leq K$ the lower part, for which $\operatorname{codim} B_{i-1} \geq 3$, and $i = L+1, \ldots, K$ the upper part, for which $\operatorname{codim} B_{i-1} = 2$. It may happen that $L = K$ and the upper part is empty.

Proposition 5.5 *The discrepancy $a(E_1)$ of the blowup $\varphi_{1,0}: V_1 \to V$, centred at B_0, is $\delta_1 = \operatorname{codim} B_0 - 2$.*

Proof From the proof of proposition 5.1, we have that

$$K_{V_1} = -\pi^*(H) + (k-2)E_1.$$

Hence, $\delta_1 = k - 2$, as required. □

On the set of exceptional divisors $\{E_1, \ldots, E_K\}$, we introduce an oriented graph structure, as described in section 4.3, as follows. Let $i \to j$ if $i > j$ and $B_{i-1} \subset E_j^{i-1}$ (strict transform of E_j on V_{i-1}). Let $p_{i,j}$ denote the number of paths from E_i to E_j for $i > j$ in the oriented graph just described, and set $p_{i,i} = 1$. Set $\delta_1 = \operatorname{codim} B_0 - 2$, and $\delta_j = \operatorname{codim} B_{j-1} - 1$ for $j = 2, \ldots, K$, and let $\nu_j = \operatorname{ord}_{E_j} \varphi_{j,j-1}^*(\Sigma^{j-1})$.

Proposition 5.6 *For $i = 1 \ldots K$,* (i) *the multiplicity of the linear system Σ at E_i is $\nu_{E_i}(\Sigma) = \sum_{j=1}^{i} p_{i,j}\nu_j$, and* (ii) *the discrepancy $K(V, \nu_{E_i})$ of ν_{E_i} is $a(E_i) = \sum_{j=1}^{i} p_{i,j}\delta_j$.*

Proof (i) Let D be a generic divisor in Σ. Clearly, $\varphi_{i,0}^*(D) = \varphi_{i,i-1}^*(\varphi_{i-1,0}^*(D))$, where from the definition, $\varphi_{i-1,0}^*(D) = D^{i-1} + \nu_{E_1}(D)E_1^{i-1} + \ldots + \nu_{E_{i-1}}(D)E_{i-1}$.

Hence,

$$\begin{aligned} \varphi_{i,0}^*(D) &= \varphi_{i,i-1}^*(D^{i-1}) + \nu_{E_1}(D)\varphi_{i,i-1}^*(E_1^{i-1}) + \ldots + \nu_{E_{i-1}}(D)\varphi_{i,i-1}^*(E_{i-1}) \\ &= D^i + \nu_i E_i + \nu_{E_1}(D)\varphi_{i,i-1}^*(E_1^{i-1}) + \ldots + \nu_{E_{i-1}}(D)\varphi_{i,i-1}^*(E_{i-1}). \end{aligned}$$

Now, since all the exceptional divisors are smooth,

$$\varphi_{i,i-1}^*(E_j^{i-1}) = \begin{cases} E_j^i & \text{if } i \not\to j \qquad (B_{i-1} \not\subset E_j^{i-1}) \\ E_j^i + E_i & \text{if } i \to j \qquad (B_{i-1} \subset E_j^{i-1}) \end{cases}$$

Hence, $\nu_{E_i}(D) = \nu_i + \sum_{i \to j} \nu_{E_j}(D)$.

We now proceed by induction on l: $\nu_{E_l}(\Sigma) = \sum_{j=1}^{l} p_{l,j}\nu_j$ is clearly satisfied for $l = 1$, and we assume it holds for $l = 2\dots, i-1$. Suppose i has edges with $r_1 = i-1, r_2, \dots r_N$, $N \leq i-1$. Then

$$\begin{aligned} \nu_{E_i}(D) &= \nu_{E_{i-1}}(D) + \nu_{E_{r_2}}(D) + \dots + \nu_{E_{r_N}}(D) + \nu_i \\ &= \sum_{j=1}^{i-1} p_{i-1,j}\nu_j + \sum_{j=1}^{r_2} p_{r_2,j}\nu_j + \dots + \sum_{j=1}^{r_N} p_{r_N,j}\nu_j + \nu_i. \end{aligned} \tag{28}$$

Clearly, $p_{i,j} = p_{i-1,j} + p_{r_2,j} + \dots + p_{r_N,j}$. Hence,

$$\begin{aligned} \sum_{j=1}^{i-1} p_{i-1,j}\nu_j &= \sum_{j=1}^{i-1} (p_{i,j} - p_{r_2,j} - \dots - p_{r_N,j})\nu_j \\ &= \sum_{j=1}^{i-1} p_{i,j}\nu_j - \sum_{j=1}^{r_2} p_{r_2,j}\nu_j - \dots - \sum_{j=1}^{r_N} p_{r_N,j}\nu_j, \end{aligned}$$

where the last equality follows from the fact that $p_{r_m,j} = 0$ if $j > r_m$. Finally, it follows from (28) that

$$\nu_{E_i}(D) = \sum_{j=1}^{i-1} p_{i,j}\nu_j + \nu_i = \sum_{j=1}^{i} p_{i,j}\nu_j.$$

(ii) Clearly, we have that $K_{V_i} = \varphi_{i,i-1}^*(K_{V_{i-1}}) + \delta_i E_i$, where

$$K_{V_{i-1}} = \varphi_{i-1,0}^*(K_V) + a(E_1)E_1^{i-1} + \dots + a(E_{i-1})E_{i-1}.$$

Hence,

$$K_{V_i} = \varphi_{i,0}^*(K_V) + a(E_1)\varphi_{i,i-1}^*(E_1^{i-1}) + \dots + a(E_{i-1})\varphi_{i,i-1}^*(E_{i-1}) + \delta_i E_i.$$

The remainder of the proof is identical to (i) if we replace $\nu_{E_i}(D)$ by $a(E_i)$, and ν_i by δ_i. □

Let $Z = \sum_j m_j Y_j$, with $Y_j \subset E_1$, be an r-cycle for some $r \geq \dim B$. Define the *degree* of Z as

$$\deg Z = \sum_j m_j \deg(Y_j \cap \varphi_{1,0}^{-1}(b)),$$

where $b \in B$ is a generic point, and the degree on the right-hand side is the ordinary degree in the projective space $\varphi_{1,0}^{-1}(b) \cong \mathbb{P}^{\operatorname{codim} B-1}$. Furthermore, for generic divisors $D_1, D_2 \in \Sigma \subset |-nK_V| = |nH|$, write D_1^i, D_2^i for their strict transforms on the blowup V_i, and $\circ$ for the codimension 2 cycle of the scheme theoretic intersection.

Proposition 5.7 (i) *Assume that* $1 \le i \le L$, *so that* $\operatorname{codim} B_{i-1} \ge 3$. *Then*

$$D_1^i \circ D_2^i = (D_1^{i-1} \circ D_2^{i-1})^i + Z_i,$$

where $\operatorname{Supp} Z_i \subset E_i$. *Furthermore,*

$$\operatorname{mult}_{B_0}(D_1 \circ D_2) = 2\left(\operatorname{ord}_{E_1} \varphi_{1,0}^*(D_1)\right)\left(\operatorname{ord}_{E_1} \varphi_{1,0}^*(D_2)\right) + \deg Z_1,$$

and for $2 \le i \le L$,

$$\operatorname{mult}_{B_{i-1}}(D_1^{i-1} \circ D_2^{i-1}) = (\operatorname{mult}_{B_{i-1}} D_1^{i-1})(\operatorname{mult}_{B_{i-1}} D_2^{i-1}) + \deg Z_i.$$

(ii) *Assume that* $L+1 \le i \le K$, *so that* $\operatorname{codim} B_{i-1} = 2$. *Then*

$$D_1^i \circ D_2^i = Z_i + Z_i',$$

where $\operatorname{Supp} Z_i \subset E_i$, $\operatorname{Supp}(\varphi_{i,i-1}(Z_i'))$ *does not contain* B_{i-1}, *and*

$$D_1^{i-1} \circ D_2^{i-1} = [(\operatorname{mult}_{B_{i-1}} D_1^{i-1})(\operatorname{mult}_{B_{i-1}} D_2^{i-1}) + \deg Z_i] B_{i-1} + (\varphi_{i,i-1})_* Z_i'.$$

Proof It is clear that we have equality of $D_1^1 \circ D_2^1$ and $(D_1 \circ D_2)^1$ away from the exceptional divisor E_1; since the latter is a strict transform, it does not contain components of E_1, and hence $D_1^1 \circ D_2^1 = (D_1 \circ D_2)^1 + Z_1$.

The first step is to reduce the problem to a two dimensional computation: intersect V_0 with a generic hyperplane section of V_0 that intersects B_0. Repeat this step $\dim B_0$ times, and choose one of the points of intersection with B_0 to be b. Continue to intersect by generic hyperplane sections containing b until V_0 is a surface. Hence, for $b \in B_0$ a generic point, we get a germ $S \ni b$ of a surface in general position with B_0 (a surface with a non-degenerate quadratic point at b), and S^1 its strict transform on V_1, a nonsingular surface. Then calculating the multiplicity of the intersection $D_1 \circ D_2$ at B_0 is reduced to computing the intersection number of two curves, $D_1|_S$ and $D_2|_S$, at b in terms of its blowup. Denote the restricted divisors by D_1 and D_2, their strict transforms on S^1 by D_1^1 and D_2^1, respectively, and let $E = E_1|_{S^1}$ be the restricted exceptional divisor. Set $\nu_1 = \operatorname{ord}_E \varphi_{1,0}^*(D_i)$. Then

$$D_i^1 = \varphi_{1,0}^*(D_i) - \nu_1 E,$$

and so

$$\begin{aligned}
D_1^1 \cdot D_2^1 &= (\varphi_{1,0}^*(D_1) - \nu_1 E) \cdot (\varphi_{1,0}^*(D_2) - \nu_1 E) \\
&= \varphi_{1,0}^*(D_1) \cdot \varphi_{1,0}^*(D_2) - \nu_1 \varphi_{1,0}^*(D_1) \cdot E - \nu_1 \varphi_{1,0}^*(D_2) \cdot E + \nu_1^2 E^2 \\
&= \varphi_{1,0}^*(D_1) \cdot \varphi_{1,0}^*(D_2) + \nu_1^2 E^2.
\end{aligned}$$

Hence,

$$\begin{aligned}(D_1 \cdot D_2)_b &= (D_1^1 \cdot D_2^1)_E - \nu_1^2 E^2 \\ &= \sum_{y \in D_1^1 \cap D_2^1 \cap E} (D_1^1 \cdot D_2^1)_y - \nu_1^2 E^2.\end{aligned}$$

Let $H \ni b$ be the class of a hyperplane section in S. Then

$$\begin{aligned}H^1 &= \varphi_{1,0}^*(H) - E \\ &\sim H - E,\end{aligned}$$

where H^1 is the class of a hyperplane section in S^1, intersecting E. Now, due to the quadratic singularity, the exceptional divisor E has degree 2, and since $(H-E)\cdot E = -E^2$, we have the self intersection $E^2 = -2$, and so

$$\operatorname{mult}_{B_0}(D_1 \circ D_2) = 2\nu_1^2 + \deg Z_1$$

as required. Note that for $i \geq 1$, V_i is smooth, and so the exceptional divisor in the blowup will have degree 1.

□

Remark 5.8 Note that $\operatorname{mult}_{B_0} \Sigma = 2 \operatorname{ord}_{E_1} \varphi_{1,0}^*(\Sigma)$, and $\operatorname{mult}_{B_{j-1}} \Sigma^{j-1} = \operatorname{ord}_{E_j} \varphi_{j,j-1}^*(\Sigma^{j-1})$, for $j = 2, \ldots, K$.

Let $D_1, D_2 \in \Sigma$ be generic divisors, set $D_1 \circ D_2 = Z_0$, and thus define a sequence of codimension 2 cycles on the blowups V_i, with

$$D_1^i \circ D_2^i = (D_1^{i-1} \circ D_2^{i-1})^i + Z_i,$$

where $Z_i \subset E_i$, so that for any $i \leq L$, we have

$$D_1^{i-1} \circ D_2^{i-1} = Z_0^{i-1} + Z_1^{i-1} + \ldots + Z_{i-2}^{i-1} + Z_{i-1}. \tag{29}$$

For any j with $i < j \leq L$, set

$$m_{i,j} = \operatorname{mult}_{B_{j-1}}(Z_i^{j-1}),$$

where for an arbitrary cycle, we extend the definition of multiplicity of an irreducible cycle, along a smaller cycle, by linearity.

Lemma 5.9 *If $m_{i,j} > 0$, then $j \to i$.*

Proof Recall that $j \to i$ if $j > i$ and $B_{j-1} \subset E_i^{j-1}$. So, for $m_{i,j} > 0$, some component of Z_i^{j-1} contains B_{j-1}. But $Z_i \subset E_i$ implies $Z_i^{j-1} \subset E_i^{j-1}$, and so $B_{j-1} \subset E_i^{j-1}$. □

Set $d_i = \deg Z_i$.

Lemma 5.10 *For* $1 \leq i < j \leq L$, *we have*

$$m_{i,j} \leq d_i.$$

Proof Since $\varphi_{i,j}(B_i) = B_j$ for $i \geq j$, we can count multiplicities at generic points. Now, the multiplicities are non-increasing with respect to the blowup, and so it remains to show that $\mathrm{mult}_{B_i} Z_i \leq \deg Z_i$. Consider a fibre over a generic point $b_i \in B_i$ of the blowup $\varphi_{i+1,i} : V_{i+1} \to V_i$, so that Z_i restricted to the fibre is a hypersurface in a projective space. Clearly, $\deg Z_i \geq \mathrm{mult}_{b_i} Z_i$, and so

$$\deg Z_i \geq \mathrm{mult}_{b_i} Z_i \geq \mathrm{mult}_{b_{j-1}}(Z_i^{j-1}) = m_{i,j}.$$

□

The following set of equalities are obtained by inserting (29) into proposition 5.7(i):

$$\left.\begin{array}{rcl} 2\nu_1^2 + d_1 & = & m_{0,1}, \\ \nu_2^2 + d_2 & = & m_{0,2} + m_{1,2}, \\ & \vdots & \\ \nu_i^2 + d_i & = & m_{0,i} + \ldots + m_{i-1,i}, \\ & \vdots & \\ \nu_L^2 + d_L & = & m_{0,L} + \ldots + m_{L-1,L}. \end{array}\right\} \qquad (30)$$

Now, carrying forward the notation,

$$\begin{aligned} d_L &\geq \nu_{L+1}^2 + \deg Z_{L+1} \\ &\geq \nu_{L+1}^2 + [\deg((\varphi_{L+1,L})_* B_{L+1})](D_1^{L+1} \cdot D_2^{L+1})_{b_{L+1}} \\ &= \nu_{L+1}^2 + [\deg((\varphi_{L+1,L})_* B_{L+1})] \left(\nu_{L+2}^2 + [\deg(\varphi_{L+2,L+1})_*(B_{L+2})](D_1^{L+2} \cdot D_2^{L+2})_{b_{L+2}}\right) \\ &= \nu_{L+1}^2 + [\deg((\varphi_{L+1,L})_* B_{L+1})]\nu_{L+2}^2 + [\deg(\varphi_{L+2,L})_*(B_{L+2})](D_1^{L+2} \cdot D_2^{L+2})_{b_{L+2}} \\ &\vdots \\ &\geq \sum_{i=L+1}^{K} \nu_i^2 (\deg(\varphi_{i-1,L})_* B_{i-1}) \\ &\geq \sum_{i=L+1}^{K} \nu_i^2. \end{aligned} \qquad (31)$$

Definition 5.11 A function $a : \{1, \ldots, L\} \to \mathbb{R}_+$ is said to be ***compatible with the graph structure*** if

$$a(i) \geq \sum_{j \to i} a(j)$$

for any $i = 1, \ldots, L$.

In particular, it is clear that $p_{K,i}$ is compatible with the graph structure. Set $a(i) = p_{K,i}$.

Theorem 5.12

$$\sum_{i=1}^{L} p_{K,i} m_{0,i} \geq p_{K,1}\nu_1^2 + \sum_{i=1}^{L} p_{K,i}\nu_i^2 + p_{K,L} \sum_{i=L+1}^{K} \nu_i^2.$$

Proof Firstly, multiply the ith equality in (30) by $p_{K,i}$, and add the equations by equating the sum of the rows on the left-hand side with the sum of the columns on the right-hand side:

$$2p_{K,1}\nu_1^2 + \sum_{i=2}^{L} p_{K,i}\nu_i^2 + \sum_{i=1}^{L-1} p_{K,i}d_i + p_{K,L}d_L = \sum_{i=1}^{L} p_{K,i}m_{0,i} + \sum_{i=2}^{L} p_{K,i}m_{1,i} + \ldots + p_{K,L}m_{L-1,L}.$$

But for $j = 1, \ldots, L$,

$$\sum_{i=j+1}^{L} p_{K,i}m_{j,i} = \sum_{\substack{i=j+1 \\ m_{j,i} \neq 0}}^{L} p_{K,i}m_{j,i} \leq d_j \sum_{i \to j} p_{K,i} \leq p_{K,j}d_j,$$

where the first inequality follows from lemmas 5.9 and 5.10, and the second from definition 5.11. Hence,

$$\begin{aligned}
2p_{K,1}\nu_1^2 + \sum_{i=2}^{L} p_{K,i}\nu_i^2 + \sum_{i=1}^{L-1} p_{K,i}d_i + p_{K,L}d_L &\leq \sum_{i=1}^{L} p_{K,i}m_{0,i} + \sum_{i=1}^{L-1} p_{K,i}d_i \\
\iff \qquad 2p_{K,1}\nu_1^2 + \sum_{i=2}^{L} p_{K,i}\nu_i^2 + p_{K,L}d_L &\leq \sum_{i=1}^{L} p_{K,i}m_{0,i} \\
\iff \qquad p_{K,1}\nu_1^2 + \sum_{i=1}^{L} p_{K,i}\nu_i^2 + p_{K,L}d_L &\leq \sum_{i=1}^{L} p_{K,i}m_{0,i}.
\end{aligned}$$

From (31), we have that $\sum_{i=L+1}^{K} \nu_i^2 \leq d_L$, so that

$$p_{K,1}\nu_1^2 + \sum_{i=1}^{L} p_{K,i}\nu_i^2 + p_{K,L} \sum_{i=L+1}^{K} \nu_i^2 \leq p_{K,1}\nu_1^2 + \sum_{i=1}^{L} p_{K,i}\nu_i^2 + p_{K,L}d_L \leq \sum_{i=1}^{L} p_{K,i}m_{0,i}$$

□

Corollary 5.13 *Set* $m = m_{0,1} = \mathrm{mult}_{B_0}(D_1 \circ D_2)$, $D_i \in \Sigma$. *Then*

$$m\sum_{i=1}^{L} p_{K,i} \geq p_{K,1}\nu_1^2 + \sum_{i=1}^{K} p_{K,i}\nu_i^2.$$

Proof We have

$$m\sum_{i=1}^{L} p_{K,i} \geq \sum_{i=1}^{L} p_{K,i}m_{0,i} \geq p_{K,1}\nu_1^2 + \sum_{i=1}^{L} p_{K,i}\nu_i^2 + p_{K,L}\sum_{i=L+1}^{K} \nu_i^2,$$

where the first inequality follows from the fact that the multiplicities, $m_{0,i}$, are non-increasing with respect to the blowup, and the second follows from theorem 5.12. Furthermore,

$$p_{K,L}\sum_{i=L+1}^{K} \nu_i^2 \geq \sum_{i=L+1}^{K} p_{K,i}\nu_i^2,$$

since it is clear that $p_{K,L} \geq p_{K,i}$ for $i = L+1, \ldots, K$, and so

$$m\sum_{i=1}^{L} p_{K,i} \geq p_{K,1}\nu_1^2 + \sum_{i=1}^{L} p_{K,i}\nu_i^2 + \sum_{i=L+1}^{K} p_{K,i}\nu_i^2 = p_{K,1}\nu_1^2 + \sum_{i=1}^{K} p_{K,i}\nu_i^2.$$

□

Corollary 5.14 *Set* $p_i = p_{K,i}$. *Then*

$$m\left(\sum_{i=1}^{L} p_i\right)\left(p_1 + 2\sum_{i=2}^{K} p_i\right) > 2n^2\left(p_1 + 2\sum_{i=2}^{L} p_i + \sum_{i=L+1}^{K} p_i\right)^2.$$

In particular, $m > 2n^2$.

Proof From proposition 5.5, the discrepancy $a(E_1)$ of the first blowup is $\mathrm{codim}\, B_0 - 2 = k-2$ due to the quadratic singularity, rather than $\mathrm{codim}\, B_0 - 1$. Setting $\delta_1 = \mathrm{codim}\, B_0 - 2$, and $\delta_i = \mathrm{codim}\, B_{i-1} - 1$ for $i = 2, \ldots K$, the Noether-Fano inequality can be rewritten

$$\sum_{i=1}^{K} p_i\nu_i > n\left(\sum_{i=1}^{K} p_i\delta_i\right).$$

From corollary 5.13, we have

$$m\sum_{i=1}^{L} p_i \geq 2p_1\nu_1^2 + \sum_{i=2}^{K} p_i\nu_i^2.$$

We now use Lagrange multipliers to find the minimum of $2p_1\nu_1^2 + \sum_{i=2}^{K} p_i\nu_i^2$:

Set $F(\nu_1, \ldots, \nu_K) = 2p_1\nu_1^2 + \sum_{i=2}^{K} p_i\nu_i^2$, with minimizing condition $G := \sum_{i=1}^{K} p_i\nu_i - n\sum_{i=1}^{K} p_i\delta_i = 0$. Define auxiliary function $\Lambda(\nu_1, \ldots, \nu_K, \lambda) = 2p_1\nu_1^2 + \sum_{i=2}^{K} p_i\nu_i^2 + \lambda\left(\sum_{i=1}^{K} p_i\nu_i - n\sum_{i=1}^{K} p_i\delta_i\right)$, and compute the partial derivatives:

$$\begin{aligned}
\frac{\partial\Lambda}{\partial\nu_1} &= 4p_1\nu_1 + \lambda p_1 \\
\frac{\partial\Lambda}{\partial\nu_i} &= 2p_i\nu_i + \lambda p_i \qquad\qquad i = 2, \ldots, K \\
\frac{\partial\Lambda}{\partial\lambda} &= \sum_{i=1}^{K} p_i\nu_i - n\sum_{i=1}^{K} p_i\delta_i.
\end{aligned}$$

Since $p_i \neq 0$, solving $\nabla_{\nu_1,\ldots,\nu_K,\lambda}\Lambda(\nu_1, \ldots, \nu_K, \lambda) = 0$ yields

$$\nu_1^* = -\frac{\lambda}{4}, \qquad \nu_i^* = -\frac{\lambda}{2} \quad (i = 2, \ldots, K),$$

and substituting these into $\frac{\partial\Lambda}{\partial\lambda}$ gives

$$-\frac{\lambda}{4}p_1 - \frac{\lambda}{2}\sum_{i=2}^{K} p_i - n\sum_{i=1}^{K} p_i\delta_i = 0 \Longleftrightarrow \lambda = -4n\left(\frac{\sum_{i=1}^{K} p_i\delta_i}{p_1 + 2\sum_{i=2}^{K} p_i}\right).$$

Hence,

$$F(\nu_1^*, \ldots, \nu_K^*) = 2p_1n^2\left(\frac{\left[\sum_{i=1}^{K} p_i\delta_i\right]^2}{\left[p_1 + 2\sum_{i=2}^{K} p_i\right]^2}\right) + 4n^2\sum_{i=2}^{K} p_i\left(\frac{\left[\sum_{j=1}^{K} p_j\delta_j\right]^2}{\left[p_1 + 2\sum_{j=2}^{K} p_j\right]^2}\right)$$

$$= 2n^2\left(\frac{\left[\sum_{i=1}^{K} p_i\delta_i\right]^2}{p_1 + 2\sum_{i=2}^{K} p_i}\right)$$

Since $F(\nu_1, \ldots, \nu_K) > F(\nu_1^*, \ldots, \nu_K^*)$,

$$\left(2p_1\nu_1^2 + \sum_{i=2}^{K} p_i\nu_i^2\right)\left(p_1 + 2\sum_{i=2}^{K} p_i\right) > 2n^2\left(\sum_{i=1}^{K} p_i\delta_i\right)^2,$$

and corollary 5.13 gives

$$m\left(\sum_{i=1}^{L} p_i\right)\left(p_1+2\sum_{i=2}^{K} p_i\right) \geq \left(2p_1\nu_1^2+\sum_{i=2}^{K} p_i\nu_i^2\right)\left(p_1+2\sum_{i=2}^{K} p_i\right)$$
$$> 2n^2\left(\sum_{i=1}^{K} p_i\delta_i\right)^2.$$

For $\dim V = m \geq 3$, $\delta_1 \geq 1$, $\delta_i \geq 2$ for $i = 2, \ldots, L$, and $\delta_i = 1$ for $i = L+1, \ldots, K$. Hence,

$$m\left(\sum_{i=1}^{L} p_i\right)\left(p_1+2\sum_{i=2}^{K} p_i\right) > 2n^2\left(p_1+2\sum_{i=2}^{L} p_i+\sum_{i=L+1}^{K} p_i\right)^2. \tag{32}$$

Set $A = \sum_{i=2}^{L} p_i$ and $B = \sum_{i=L+1}^{K} p_i$. Then (32) can be rewritten as

$$m(p_1+A)(p_1+2A+2B) > 2n^2(p_1+2A+B)^2,$$

and we are done since

$$\begin{aligned} & \frac{(p_1+A)(p_1+2A+2B)}{(p_1+2A+B)^2} && \leq 1 \\ \iff\ & p_1A+2A^2+2AB+B^2 && \geq 0 \end{aligned}$$

□

Proposition 5.15 *There exists no maximal singularity with centre $B_0 = P^*$.*

Proof In the formulation of the main result (section 5.1), a linear subspace $P \subset \mathbb{P}^m$ of codimension k was fixed, satisfying the inequality $\dim V = m \leq \frac{k(k-3)}{2} + 2$, for $m \geq 6$. That is to say, $\operatorname{codim} P^* \geq 5$. In fact, for the case $B_0 = P^*$, the result holds for $\operatorname{codim} B_0 = k \geq 3$.

First, we exclude the case of B_0 being the centre of a maximal subvariety of the movable linear system $\Sigma \subset |-nK_V|$. Assume the opposite. Let $D_1, D_2 \in \Sigma$ be generic divisors, with $m_{0,1} = \operatorname{mult}_{B_0}(D_1 \circ D_2)$. By proposition 5.7, we have that

$$\operatorname{mult}_{B_0}(D_1 \circ D_2) \geq 2\nu_1^2,$$

where $\nu_1 = \operatorname{ord}_{E_1} \varphi_{1,0}^*(D_i)$. Now, if B_0 is a maximal subvariety, so that $\operatorname{codim} B_0 = \ldots = \operatorname{codim} B_{K-1}$, then the Noether-Fano inequality holds,

$$\sum_{i=1}^{K} p_i(\nu_i - n\delta_i) > 0. \tag{33}$$

Moreover, since multiplicities are non-increasing with respect to the blowup, so that $2\nu_1 \geq \nu_2 \geq \ldots \geq \nu_K$, and $k \geq 3$, it follows from the Noether-Fano inequality that either $\nu_1 > n\delta_1 = n(k-2) \geq n$, or there exists an $i > 1$ such that $\nu_i > n\delta_i$. But then

$$2\nu_1 \geq \nu_i > n\delta_i = n(k-1) \geq 2n,$$

so that $\nu_1 > n$ again. Hence,

$$m_{0,1} \geq 2\nu_1^2 > 2n^2.$$

However, the anticanonical degree of the cycle $D_1 \circ D_2$ is

$$\deg(D_1 \circ D_2) = \left((D_1 \circ D_2) \cdot (-K_V)^{\dim V - 2}\right) = 2n^2,$$

where $K_V = -\pi^* H$, for H the class of a hyperplane in $\mathbb{P}^m$, since V is a double cover and $D_i \sim n\pi^* H$. Therefore, $\mathrm{mult}_{B_0}(D_1 \circ D_2) > \deg(D_1 \circ D_2)$, a contradiction.

We now consider the infinitely near case. From corollary 5.14, we have that $m_{0,1} > 2n^2$, and so a contradiction is again deduced as above. □

Remark 5.16 Although the inequality $m > 2n^2$ from corollary 5.14 was sufficient to exclude infinitely near maximal singularities, it can actually be strengthened as follows. Firstly, we introduce a modified graph structure:

$$\text{let} \quad \begin{cases} i \to j & \text{if} \quad i > j \geq 2 \qquad \text{and} \quad B_{i-1} \subset E_j^{i-1} \\ i \to 1 & \text{if} \quad L \geq i > 1 \qquad \text{and} \quad B_{i-1} \subset E_1^{i-1} \\ i \nrightarrow 1 & \text{if} \quad i \geq L+1 > 1 \quad \text{and} \quad B_{i-1} \subset E_1^{i-1} \\ i \nrightarrow j & \text{otherwise,} \end{cases}$$

and denote the number of paths from E_i to E_j, for $i > j$, by $\widetilde{p}_{i,j}$ in this modified graph, setting $\widetilde{p}_{i,i} = 1$. This structure is identical to the original graph, except that we remove all arrows $i \to 1$, where $i \in \{L+1, \ldots, K\}$. Since the graph corresponding to the lower part of the resolution remains unchanged, the function $a(i) = \widetilde{p}_{K,i}$ remains compatible with the graph structure. Furthermore, the proof of theorem 5.12 relies on lemma 5.9 for when $1 \leq i < j \leq L$ only, so for the same reason as just discussed, the lemma and theorem hold true for modified graph. It remains to show that the Noether-Fano inequality still holds.

Let $a_1 > \ldots > a_r$ be vertices, each with an arrow $K \to a_i$, and $b_1 > \ldots > b_s > L$ be vertices, each with an arrow $b_i \to 1$. We can then rewrite the Noether-Fano inequality for the original graph structure as follows

$$p_{K,K}(\nu_K - n\delta_K) + \sum_{K \to a_i} \sum_{j=1}^{K-1} p_{a_i,j}(\nu_j - n\delta_j) > 0,$$

recalling that $p_{i,j} = 0$ for $j > i$. Since we are only removing arrows $i \to 1$ for $i > L$, the above Noether-Fano inequality remains the same except for when $j = 1$. We have

$$\widetilde{p}_{a_i,1} = p_{a_i,1} - \sum_{l=1}^{s} p_{a_i,b_l}, \quad \text{for} \quad i = 1, \dots, r,$$

and so

$$\sum_{K\to a_i} \widetilde{p}_{a_i,1}(\nu_1 - n\delta_1) = \sum_{K\to a_i} \left[\left(p_{a_i,1} - \sum_{l=1}^{s} p_{a_i,b_l} \right) (\nu_1 - n\delta_1) \right].$$

But $\nu_1 \leq n\delta_1$ (else E_1 would itself be a maximal subvariety), and so

$$\sum_{K\to a_i} \widetilde{p}_{a_i,1}(\nu_1 - n\delta_1) \geq \sum_{K\to a_i} p_{a_i,1}(\nu_1 - n\delta_1).$$

Hence,

$$\sum_{i=1}^{K} \widetilde{p}_i(\nu_i - n\delta_i) > \sum_{i=1}^{K} p_i(\nu_i - n\delta_i) > 0,$$

and so the Noether-Fano inequality holds true for the modified graph structure.

By construction, there are no arrows $i \to 1$ for $i > L$ in the modified graph, and so it follows that

$$\widetilde{p}_1 = \widetilde{p}_{K,1} = \sum_{j\to 1} \widetilde{p}_{K,j} \leq \sum_{i=2}^{L} \widetilde{p}_{K,i} = \sum_{i=2}^{L} p_{K,i} = A. \tag{34}$$

We now claim that

$$2n^2(\widetilde{p}_1 + 2A + B)^2 \geq 3n^2(\widetilde{p}_1 + A)(\widetilde{p}_1 + 2A + 2B).$$

After multiplying out and cancelling terms, the inequality is reduced to

$$2A^2 + 2B^2 + 2AB \geq \widetilde{p}_1^2 + \widetilde{p}_1 A + 2\widetilde{p}_1 B,$$

which holds true by (34). Hence, by corollary 5.14,

$$m(\widetilde{p}_1 + A)(\widetilde{p}_1 + 2A + 2B) > 2n^2(\widetilde{p}_1 + 2A + B)^2 \geq 3n^2(\widetilde{p}_1 + A)(\widetilde{p}_1 + 2A + 2B),$$

giving the stronger inequality,

$$m > 3n^2.$$

6 Main theorem: the general case

In section 6.2, we explain why we cannot use the approach of the preceding chapter in the general case of the main theorem. The proof that appears in this chapter relies on *inversion of adjunction*, which follows from the *connectedness principle*, so we start by stating and proving this.

6.1 The connectedness principle of Shokurov and Kollár

6.1.1 The connectedness principle

Definition 6.1 Let Y be a closed subscheme in X, and let $i : Y \to X$ be the inclusion morphism. We define the ***ideal sheaf*** of Y, denoted $\mathcal{I}_Y$, to be the kernel of the morphism $i^{\#} : \mathcal{O}_X \to i_*\mathcal{O}_Y$. Hence, there is a short exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0,$$

where $\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{I}_Y$.

Now, let D be an effective Cartier divisor on X, and Y be the associated locally principal closed subscheme. It follows that $\mathcal{I}_Y \cong \mathcal{O}_X(-D)$, and hence there is a short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

writing $\mathcal{O}_D$ for $\mathcal{O}_X/\mathcal{O}_X(-D)$.

The proof of proposition (6.3) makes use of the following theorem:

Theorem 6.2 (Kawamata-Viehweg vanishing theorem) *Let Y be a smooth projective variety, $R = \sum_{i \in I_R} r_i R_i$ an effective $\mathbb{Q}$-divisor with $0 < r_i < 1$, R_i pair-wise distinct, where the divisor* $\operatorname{Supp} R$ *has normal crossings. Assume that $L \in \operatorname{Pic} Y$ is a class such that the class $(L - R)$ is numerically effective and big, that is, $(L - R)^{\dim Y} > 0$. Then*

$$H^j(Y, K_Y + L) = 0$$

for $j \geq 1$.

Proof See [28, 29, 30] for a proof and explanations. □

Proposition 6.3 *Let X be a normal variety,*

$$D = \sum_{i \in I_D} d_i D_i$$

an effective $\mathbb{Q}$-divisor on X ($d_i \in \mathbb{Q}^+$ for all $i \in I_D$), where the prime divisors D_i are pair-wise distinct, and $f : \widetilde{X} \to X$ be a resolution of singularities of the pair (X, D), with $\{E_i | i \in I_f\}$ the set of exceptional divisors. Let $\widetilde{D}$ be the strict transform of D on $\widetilde{X}$, where the divisor

$$\operatorname{Supp} \widetilde{D} \cup \bigcup_{i \in I_f} E_i$$

has normal crossings on $\widetilde{X}$. Set $J = I_D \cup I_f$, and write

$$K_{\widetilde{X}} = f^*(K_X + D) + \sum_{j \in J} e_j E_j, \tag{35}$$

where the $E_j \subset \widetilde{X}$ are prime divisors (exceptional or components of the strict transform $\widetilde{D}$). Assume that the class $-(K_X + D)$ is numerically effective and big. Then

$$\bigcup_{e_j \le -1} E_j$$

is connected.

Proof For each $i \in J$, set

$$m_i = \lceil e_i \rceil, \quad \alpha_i = m_i - e_i \ge 0,$$

so that $e_i = m_i - \alpha_i$, $\alpha_i < 1$, and rewrite (35) as

$$-f^*(K_X + D) = -K_{\widetilde{X}} + E^+ - E^- - \sum_{j \in J} \alpha_j E_j,$$

for effective divisors

$$E^+ = \sum_{e_i > -1} m_i E_i = \sum_{e_i > 0} m_i E_i$$

and

$$E^- = - \sum_{e_i \le -1} m_i E_i.$$

Now, $-f^*(K_X + D)$ being nef and big follows from our assumption on $-(K_X + D)$, and so applying the vanishing theorem to

$$L = -K_{\widetilde{X}} + E^+ - E^- \quad \text{and} \quad R = \sum_{j \in J} \alpha_j E_j,$$

we get

$$H^i(\widetilde{X}, E^+ - E^-) = 0 \tag{36}$$

for $i \geq 1$. Now, consider the short exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(E^+ - E^-) \to \mathcal{O}_{\widetilde{X}}(E^+) \to \mathcal{O}_{\widetilde{X}}(E^+)|_{E^-} \to 0,$$

which gives the long exact sequence

$$0 \to H^0(\widetilde{X}, E^+ - E^-) \to H^0(\widetilde{X}, E^+) \to H^0(E^-, E^+|_{E^-}) \to H^1(\widetilde{X}, E^+ - E^-) \to \ldots.$$

Applying (36) to this sequence, we conclude that the map

$$H^0(\widetilde{X}, E^+) \to H^0(E^-, E^+|_{E^-}) \tag{37}$$

is surjective.

Since E^+ is effective, each component must be f-exceptional, and so E^+ is f-exceptional. By proposition 2.10, the linear system $|E^+|$ consists of just one divisor, and so

$$H^0(\widetilde{X}, E^+) \cong \mathbb{C}.$$

By construction, E^+ and E^- are supported by divisors from disjoint sets, and so the restriction $E^+|_{E^-}$ is effective on each connected component of E^-.

Let $l \geq 1$ be the number of connected components of $\operatorname{Supp} E^-$. Then

$$H^0(E^-, E^+|_{E^-}) = \bigoplus_{\alpha=1}^{l} H^0(E^-_\alpha, E^+|_{E^-_\alpha}),$$

where E^-_α are the connected components of E^-, and so $\dim H^0(E^-, E^+|_{E^-}) \geq l$. Hence, (37) gives a surjective map

$$\mathbb{C} \to \mathbb{C}^l \to 0$$

and we conclude that $l \leq 1$, and so $l = 1$. That is to say, $\operatorname{Supp} E^-$ is connected. □

6.1.2 Inversion of adjunction

We state a particular case of inversion of adjunction here, which follows from the connectedness principle. For the general statement, see [12, chapter 17].

Proposition 6.4 *Let $x \in X$ be a factorial germ, and D an effective $\mathbb{Q}$-divisor with $x \in \operatorname{Supp} D$. Assume that the point x is an isolated centre of a non-canonical singularity of the pair (X, D), so that (X, D) is not canonical, but its restriction to $X \setminus \{x\}$ is. Let R be a divisor with $x \in R$, $R \not\subset \operatorname{Supp} D$. Then the pair $(R, D|_R)$ is not log canonical at x.*

The proof follows [12, chapter 17], in the style of [13].

Proof Let $D = \sum_{i\in I} d_i D_i$, $d_i \in \mathbb{Q}_+$. By definition, since the pair (X, D) is canonical outside x, we must have that $d_i \leq 1$ for all $i \in I$. Furthermore, we may replace D by $\frac{1}{1+\varepsilon}D$ for a small $\varepsilon \in \mathbb{Q}_+$, and so we may assume that $d_i < 1$ for all $i \in I$.

Let $\lambda : X^+ \to X$ be the blowup of the point x, $E = \lambda^{-1}(x) \subset X^+$ the exceptional divisor, and define D^+, R^+ to be the strict transforms of D, R on X^+, respectively. Let $\mu : \widetilde{X} \to X^+$ be a resolution of singularities of the pair $(X^+, D^+ + R^+)$, and let $\varphi = \lambda \circ \mu : \widetilde{X} \to X$ be the composite map, so that $\widetilde{D}_i$, $\widetilde{R}$ are the strict transforms of D_i, R on $\widetilde{X}$, respectively. Then

$$K_{\widetilde{X}} = \varphi^*(K_X + D + R) + \sum_{j\in J} e_j E_j - \sum_{i\in I} d_i \widetilde{D}_i - \widetilde{R}, \tag{38}$$

where the E_j $(j \in J)$ are exceptional divisors of the morphism φ. By definition, we have

$$\varphi^*(D) = \sum_{i\in I} d_i \widetilde{D}_i + \sum_{j\in J} (\operatorname{ord}_{E_j} \varphi^*(D)) E_j, \quad \text{and} \quad \varphi^*(R) = \widetilde{R} + \sum_{j\in J} (\operatorname{ord}_{E_j} \varphi^*(R)) E_j.$$

From (38) we have that $e_j = a(E_j, X) - \operatorname{ord}_{E_j} \varphi^*(D) - \operatorname{ord}_{E_j} \varphi^*(R)$ for $j \in J$. Now, since not all exceptional divisors in the resolution lie over x, there is a subset $J^+ \subset J$ with

$$\varphi^{-1}(x) = \bigcup_{j\in J^+} E_j.$$

Since X is factorial,

$$\operatorname{ord}_{E_j} \varphi^*(R) \geq 1$$

for $j \in J^+$. By assumption, the pair (X, D) is not canonical so that $\nu_{E_l}(D) = \operatorname{ord}_{E_l} \varphi^*(D) > a(E_l, X)$, where $l \in J^+$ since the pair is canonical outside x. Hence

$$e_l = a(E_l, X) - \operatorname{ord}_{E_l} \varphi^*(D) - \operatorname{ord}_{E_l} \varphi^*(R) < -1. \tag{39}$$

We see from (38) that $\widetilde{R}$ appears with coefficient -1, and so the connectedness principle ensures

$$E_l \cap \widetilde{R} \neq \emptyset. \tag{40}$$

Let $\varphi_{\widetilde{R}} = \varphi|_{\widetilde{R}} : \widetilde{R} \to R$ be the restriction onto R of the sequence of blowups φ, and let K_R and D_R be the restrictions of K_X and D onto R. Then the adjunction formula gives $K_{\widetilde{R}} = (K_{\widetilde{X}} + \widetilde{R})|_{\widetilde{R}}$ and $K_R = (K_X + R)|_R$; applying this to (38), we obtain

$$\begin{aligned} K_{\widetilde{R}} &= (K_{\widetilde{X}} + \widetilde{R})|_{\widetilde{R}} \\ &= (\varphi^*(K_X + D + R))|_{\widetilde{R}} + \sum_{j\in J} e_j E_j|_{\widetilde{R}} - \sum_{i\in I} d_i \widetilde{D}_i|_{\widetilde{R}} \\ &= \varphi_{\widetilde{R}}^*(K_R + D_R) + \sum_{j\in J} e_j E_j|_{\widetilde{R}} - \sum_{i\in I} d_i \widetilde{D}_i|_{\widetilde{R}} \end{aligned}$$

From (39) and (40), we see that there is an $l \in J^+$ such that $E_l|_{\widetilde{R}}$ has coefficient strictly less than -1. It follows that

$$\begin{aligned} a(E_l, R) &= e_l + \operatorname{ord}_{E_l} \varphi^*_{\widetilde{R}}(D_R) \\ &< \operatorname{ord}_{E_l} \varphi^*_{\widetilde{R}}(D_R) - 1. \end{aligned}$$

That is to say, $\operatorname{ord}_{E_l} \varphi^*_{\widetilde{R}}(D_R) > a(E_l, R) + 1$, and so the pair (R, D_R) is not log canonical at x. □

6.2 The general case

Unfortunately, we cannot use the approach of the preceding chapter when $B \subset P^*$ is an arbitrary subvariety. The reason is that when blowing up B and then the centre of the maximal singularity on the blowup and so on, we obtain a sequence of singular varieties, the singularities of which are difficult to control. We will therefore use an alternative approach.

Firstly, let $\Pi \subset \mathbb{P}^m$ be a generic linear subspace of dimension $\operatorname{codim} B \geq 6$, $V_\Pi = \pi^{-1}(\Pi)$ the corresponding double cover, and $\Sigma_\Pi = \Sigma|_{V_\Pi}$ a movable linear system defined as the restriction of the linear system Σ onto V_Π. By genericity of Π, the pair $(V_\Pi, \frac{1}{n}\Sigma_\Pi)$ is not canonical, that is, Σ_Π has a maximal singularity, the centre of which is a point in the singular locus $P^* \cap V_\Pi$; namely, any of the points of intersection $B \cap V_\Pi$. Hence, we are reduced to the case when the centre of a maximal singularity is a point in the singular locus. In order to study this case, we need to investigate the degenerate fibres of E^+ over $P^* \cong P$.

6.3 Degenerating quadrics

Let S be the space of $k \times k$ symmetric matrices with $\mathbb{C}$-coefficients, having dimension $\frac{k(k+1)}{2}$, and $\mathbf{S}$ the space of $k \times k$ symmetric matrices with homogeneous polynomial coefficients of degree $2m-2$ in $x_k, \dots, x_m$. Since the number of monomials of degree n in r variables is $\binom{n+r-1}{n}$, it follows that $\mathbf{S}$ has dimension $\frac{k(k+1)}{2}\binom{3m-k-2}{2m-2}$.

Let (p, A) be an element of $\mathbb{P}^{m-k} \times \mathbf{S}$, where we identify $\mathbb{P}^{m-k}$ with P^*. Note that the rank $\operatorname{rk} A(p)$ does not depend on the choice of coordinates of p: when we replace $(a_k, \dots, a_m)$ by $(\lambda a_k, \dots, \lambda a_m)$, $A(a_k, \dots, a_m)$ is replaced by $\lambda^{2m-2} A(a_k, \dots, a_m)$, so the rank of the matrix remains the same, and therefore $\operatorname{rk} A(p)$ is well defined for $p \in \mathbb{P}^{m-k}$. Define $\mathbf{X}_e = \{(p, A) | \operatorname{rk} A(p) \leq k - e\} \subset \mathbb{P}^{m-k} \times \mathbf{S}$ and set $\mathbf{X}_e(A) = \{p \in \mathbb{P}^{m-k} | \operatorname{rk} A(p) \leq k - e\}$ for $A \in \mathbf{S}$ and $\mathbf{X}_e(p) = \{A | \operatorname{rk} A(p) \leq k - e\}$ for $p \in \mathbb{P}^{m-k}$.

Lemma 6.5 *If* $\frac{e(e+1)}{2} \geq m - k + 1$, *then for generic* $A \in \mathbf{S}$ *we have* $\mathbf{X}_e(A) = \emptyset$, *that is, for every* $p \in \mathbb{P}^{m-k}$,

$$\operatorname{rk} A(p) \geq k - e + 1.$$

Proof The lemma obviously follows from

Lemma 6.6 *For each* $e \geq 0$ *there is a non-empty Zariski open subset* $U \subset \mathbf{S}$ *such that for* $A \in U$,

$$\operatorname{codim}\left(\mathbf{X}_e(A) \subset \mathbb{P}^{m-k}\right) = \min\left\{\frac{e(e+1)}{2}, m - k + 1\right\}.$$

Proof of Lemma 6.6 We have $\mathbf{X}_e(A) = \mathbf{X}_e \cap \left(\mathbb{P}^{m-k} \times \{A\}\right)$, and $\mathbb{P}^{m-k} \times \{A\}$ are just fibres of the projection $\mathbb{P}^{m-k} \times \mathbf{S} \to \mathbf{S}$. We may assume surjectivity of the restricted projection $\mathbf{X}_e \to \mathbf{S}$ for the following reason: if not surjective, this means there exists $A \in \mathbf{S}$ with $\mathbf{X}_e(A) = \emptyset$. That is to say, for all $p \in \mathbb{P}^{m-k}$, $\operatorname{rk} A(p) \geq k - e + 1$, and there is nothing further to prove. Taking this observation into account, the lemma follows from

Lemma 6.7 *The following equality holds:*

$$\operatorname{codim}\left(\mathbf{X}_e \subset \mathbb{P}^{m-k} \times \mathbf{S}\right) = \frac{e(e+1)}{2}.$$

Proof of Lemma 6.7 Consider the closed algebraic set $\mathbf{X}_e(p) = \mathbf{X}_e \cap (\{p\} \times \mathbf{S})$. Thus $\dim \mathbf{X}_e = \dim \mathbf{X}_e(p) + (m - k)$, and the result follows from

Lemma 6.8 *The following equality holds:*

$$\operatorname{codim}\left(\mathbf{X}_e(p) \subset \mathbf{S}\right) = \frac{e(e+1)}{2}.$$

Proof of Lemma 6.8 Define $S_e = \{M \in S | \operatorname{rk} M \leq k - e\}$. For $p = (a_k : \ldots : a_m) \in \mathbb{P}^{m-k}$, denote $\overline{p} = (a_k, \ldots, a_m) \in \mathbb{C}^{m-k+1}$, and consider the map $\{\overline{p}\} \times \mathbf{S} \to S$, given by evaluating a matrix $A \in \mathbf{S}$ at $\overline{p}$. Clearly this map is surjective, and all fibres are of the same dimension and smooth: choose coordinates such that $p = (1 : 0 : \ldots : 0)$, with coordinates $x_k, \ldots, x_m$. Then for any $M \in S$, the inverse image is $\{\overline{p}\} \times (M x_k^{2m-2} + N)$, where $N \subset \mathbf{S}$ is the subspace of matrices whose coefficients do not include the monomial x_k^{2m-2}, and the dimension of a fibre is $\Delta = \frac{k(k+1)}{2}\left(\binom{3m-k-2}{2m-2} - 1\right)$.

Now, recalling from what was said earlier, we have that $\mathbf{X}_e(\overline{p}) = \mathbf{X}_e(p)$. Consider the restriction map $\mathbf{X}_e(\overline{p}) \to S_e$. Then, since this map is surjective, we have that

$$\dim \mathbf{X}_e(\overline{p}) = \dim S_e + \Delta.$$

That is to say,

$$\operatorname{codim}(\mathbf{X}_e(p) \subset \mathbf{S}) = \operatorname{codim}(S_e \subset S),$$

and the lemma follows from

Lemma 6.9 *The following equality holds:*

$$\operatorname{codim}(S_e \subset S) = \frac{e(e+1)}{2}.$$

Proof of Lemma 6.9 First, note that if we define $Y_i := \{M \in S |\, \operatorname{rk} M = k - i\}$, then $S_e = \bigcup_{i=e}^{k} Y_i$, so that $\dim X_e = \dim Y_e$, and hence it is enough to prove the lemma for Y_e.

Let $M \in S$ be a symmetric matrix of rank $k-e$. It is a well known result that a symmetric matrix of rank $k-e$ has at least one non-zero *principal* minor of order $k-e$, and so by swapping the relevant same indexed rows and columns of M, so that the result is still a symmetric matrix, we may assume that M has the following form:

$$M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where D is a non-degenerate symmetric $(k-e)\times(k-e)$ submatrix, $B^T = C$, and A is a symmetric $e \times e$ submatrix. Now, consider the submatrix $\left(\begin{smallmatrix} B \\ D \end{smallmatrix}\right)$. Since row rank equals column rank, this submatrix has rank $k-e$, and so each row of B is a linear combination of rows of D, where the only condition on entries in B is the initial symmetric condition on M. Similarly, for the columns of C and D. Explicitly, if $M = (a_{ij})$, then B and C have entries:

$$\begin{aligned} a_{ij} &= \sum_{l=e+1}^{k} r_l a_{lj} \quad \text{for} \quad 1 \le i \le e, \quad e+1 \le j \le k, \quad \text{and} \\ a_{ji} &= \sum_{l=e+1}^{k} r_l a_{jl} \quad \text{for} \quad 1 \le i \le e, \quad e+1 \le j \le k, \quad \text{respectively.} \end{aligned}$$

Since the last $k-e$ rows of M are linearly independent, the first e rows are linear combinations of these. In particular,

$$a_{ij} = \sum_{l=e+1}^{k} r_l a_{lj} \quad \text{for} \quad 1 \le i \le e, \quad 1 \le j \le e,$$

where the r_l are as determined above, so that the entries of A are fixed. Since this submatrix is symmetric, there are precisely $\frac{1}{2}e(e+1)$ constraints on these entries.

An alternative approach is as follows: let Q denote a quadric hypersurface in $\mathbb{P}^{k-1}$, and define $S'_e = \{Q \subset \mathbb{P}^{k-1} | \operatorname{rk} Q \leq k-e\}$. Let Q' denote a quadric hypersurface in $\mathbb{P}^{k-1-e}$, and set $T = \{Q' \subset \mathbb{P}^{k-1-e} | \operatorname{rk} Q' = k-e\}$. Now, if a singular quadric Q is defined by a homogeneous equation F of degree 2, then the vertex space of Q is the vanishing locus of the system of partial derivatives of F. It follows from Euler's homogeneous function theorem that this is indeed contained in Q, and any line joining the singular vertex space to the base is contained in Q. Now, clearly, a quadric in S'_e corresponds to a symmetric matrix M of corank e (up to multiplication by a scalar), and the matrix of partial derivatives (which is of course the same as the Jacobian of F) is just a scalar multiple of M, so also has corank e. That is to say, the number of free parameters in system of equations $\frac{\partial F}{\partial x_i} = 0$ is e, and so the vertex space of Q has projective dimension $e-1$. The base of Q then corresponds to an element of T, that is, a non-degenerate quadric hypersurface in $\mathbb{P}^{k-1-e}$. Hence,

$$\dim S'_e = \dim T + \dim \operatorname{Gr}(e,k),$$

where T can be identified with the projectivisation of the space of $(k-e) \times (k-e)$ symmetric matrices, S'_e with the projectivisation of S_e defined above, and the Grassmannian $\operatorname{Gr}(e,k)$ with the space of $(e-1)$–dimensional subspaces in $\mathbb{P}^{k-1}$. We have that

$$\begin{aligned} \dim \operatorname{Gr}(e,k) &= k(k-e) - (k-e)^2 \\ &= e(k-e). \end{aligned}$$

Hence,

$$\begin{aligned} \dim S'_e &= \frac{1}{2}(k-e)(k-e+1) - 1 + e(k-e) \\ &= \frac{1}{2}k(k+1) - 1 - \frac{1}{2}e(e+1), \end{aligned}$$

and so $\operatorname{codim}(S_e \subset S) = \frac{e(e+1)}{2}$. □

Recall that $\varphi_{E^+} : E^+ \longrightarrow P^* \cong P$ is a fibration into quadrics. Now, the elements of the space of quadratic forms, $x_{m+1}^2 = \sum_{i,j=0}^{k-1} x_i x_j P_{ij}(x_k, \ldots, x_m)$ in $x_0, \ldots, x_{k-1}, x_{m+1}$ for varying $P_{ij}(x_*)$, have corresponding $(k+1) \times (k+1)$ symmetric matrices, which are clearly in one-to-one correspondence with the elements of $\mathbf{S}$. Therefore, for a given choice of P_{ij}, that is, for a given choice of W, we have a corresponding matrix $A \in \mathbf{S}$. Hence, the subspace in P^* over which the quadric fibres have rank $\leq k+1-e$ can be identified with the space $\mathbf{X}_e(A)$, and it follows from lemmas 6.5 and 6.6 that this space is either empty, or has codimension $\frac{e(e+1)}{2}$. We assumed in

Section 5.1 that $m \leq \frac{k(k-3)}{2} + 2$. By Lemma 6.5, we now conclude that the rank of any fibre of φ_{E^+} is at least 4: suppose $\mathbf{X}_{k-3}(A) \neq \emptyset$. Then

$$\begin{aligned} \dim\left(\mathbf{X}_{k-3}(A) \subset \mathbb{P}^{m-k}\right) &= m - k - \frac{(k-3)(k-2)}{2} \\ &\leq \frac{k(k-3)}{2} + 2 - k - \frac{(k-3)(k-2)}{2} \\ &= -1. \end{aligned}$$

Hence $\mathbf{X}_{k-3}(A) = \emptyset$ as required.

6.4 Inversion of adjunction

Now let us complete the proof of our theorem. Suppose the restricted pair $(V_\Pi, \frac{1}{n}\Sigma_\Pi)$ has the point $o \in B \cap V_\Pi$ as an isolated centre of a non-canonical singularity, let $\varphi : V_\Pi^+ \to V_\Pi$ be the blowup of the point o, and let E^+ be the exceptional divisor $\varphi^{-1}(o)$, considered as a quadric hypersurface in $\mathbb{P}^l$, where $\operatorname{codim} B = l$. Furthermore, let D^+ be the strict transform of $D \in \Sigma_\Pi$ on V_Π^+. For some $\mu \in \mathbb{Z}_+$, we have that

$$D^+ \sim \varphi^* D - \mu E^+. \tag{41}$$

Consider the pair $(V_\Pi, \frac{1}{n}\Sigma_\Pi)$. We now argue as in [36], see also Section 1 in [37]. Take a generic 3-plane $\Lambda \subset \Pi$, containing the point o. Set $V_\Lambda = \pi^{-1}(\Lambda)$, and let Σ_Λ be the restriction of Σ onto V_Λ. By inversion of adjunction (proposition 6.4, or see [38, 12]), the pair $(V_\Lambda, \frac{1}{n}\Sigma_\Lambda)$ is not log canonical. However, the point $o \in V_\Lambda$ is a non-degenerate 3-fold quadratic singularity: since Λ is generic, the restriction of the quadric E onto the blowup V_Λ^+ does not reduce its rank.

We now argue as in proposition 1.3, [36] (or as in [39], proof of proposition 3.10), to conclude that $\mu > n$. Assume the opposite: $\mu \leq n$. Recall the notation from section 5.4. Let $\varphi_{1,0} : V_1 \to V_\Lambda$ be the blowup of the point o, $E_1 = E^+$ be the exceptional divisor $\varphi_{1,0}^{-1}(o)$, and let Σ_Λ^1 be the strict transform of the system Σ_Λ on $V_1 = V_\Lambda^+$. Let $E^\sharp \subset V^\sharp$ be a maximal singularity of the pair $(V_\Lambda, \frac{1}{n}\Sigma_\Lambda)$, centred at $o \in V_\Lambda$, for some model $\psi : V^\sharp \to V_\Lambda$ of V_Λ, so that the Noether-Fano inequality $\nu_{E^\sharp}(\frac{1}{n}\Sigma_\Lambda) > a(E^\sharp, V_\Lambda)$ holds, and the resolution is such that $\varphi_{K,0}^{-1} \circ \psi(E^\sharp) = E_K$.

Proposition 6.10 *The pair $(V_\Lambda^+, \frac{1}{n}\Sigma_\Lambda^+)$ is not canonical. Moreover, the centre of any non-canonical singularity of this pair is contained in the exceptional divisor E^+.*

Proof We start by making a claim.

Claim *The following equality holds:*

$$a(E_K, V_\Lambda) = a(E_K, V_1) + a(E_1, V_\Lambda)\nu_{E_K}(E_1).$$

Proof of Claim

$$\begin{aligned} K_{V_K} &= \varphi_{K,0}^*(K_{V_\Lambda}) + a(E_1, V_\Lambda)E_1^K + \ldots + a(E_K, V_\Lambda)E_K \\ K_{V_1} &= \varphi_{1,0}^*(K_{V_\Lambda}) + a(E_1, V_\Lambda)E_1. \end{aligned}$$

Hence,

$$\begin{aligned} K_{V_K} &= \varphi_{K,1}^*\left(\varphi_{1,0}^*(K_{V_\Lambda}) + a(E_1, V_\Lambda)E_1\right) + a(E_K, V_1)E_K \\ &= \varphi_{K,0}^*(K_{V_\Lambda}) + a(E_1, V_\Lambda)\varphi_{K,1}^*(E_1) + a(E_K, V_1)E_K. \end{aligned}$$

Now,

$$\varphi_{K,1}^*(E_1) = E_1^K + \nu_{E_2}(E_1)E_2^K + \ldots + \nu_{E_K}(E_1)E_K,$$

and so by comparing coefficients of E_K, the claim is proved. □

Let D^+ be the strict transform of a generic divisor $D \in \Sigma_\Lambda$ on V_Λ^+. Then

$$\frac{1}{n}\varphi_{1,0}^*(D) \sim \frac{1}{n}D^+ + \frac{\mu}{n}E^+, \tag{42}$$

and so

$$\nu_{E^\sharp}\left(\frac{1}{n}D\right) = \nu_{E^\sharp}\left(\frac{1}{n}D^+\right) + \left(\frac{\mu}{n}\right)\nu_{E^\sharp}(E^+). \tag{43}$$

Now, since $(V_\Lambda, \frac{1}{n}D)$ is not canonical, it follows from the claim and (43) that

$$\begin{aligned} \nu_{E^\sharp}\left(\frac{1}{n}D\right) &> a(E^\sharp, V_\Lambda) \\ \iff \quad \nu_{E^\sharp}\left(\frac{1}{n}D^+\right) + \left(\frac{\mu}{n}\right)\nu_{E^\sharp}(E^+) &> a(E^\sharp, V_\Lambda^+) + a(E^+, V_\Lambda)\nu_{E^\sharp}(E^+). \end{aligned}$$

Hence, either $\frac{\mu}{n} > a(E^+, V_\Lambda)$, so that $\operatorname{mult}_o\left(\frac{1}{n}D\right) = \frac{2\mu}{n} > a(E^+, V_\Lambda)$ and so o is a maximal point of $\frac{1}{n}D$, or else $\nu_{E^\sharp}\left(\frac{1}{n}D^+\right) > a(E^\sharp, V_\Lambda^+)$. □

Now, set $D_{E^+}^+ = D^+|_{E^+}$. Then by inversion of adjunction (proposition 6.4), the pair $(E^+, \frac{1}{n}D_{E^+}^+)$ is not log canonical. Let $H_{E^+} = -E^+|_{E^+}$ be the class of a hyperplane section of $E^+ \subset \mathbb{P}^3$. It follows from (42) that

$$D_{E^+}^+ \sim -\mu E^+|_{E^+} = \mu H_{E^+}.$$

However, since $E^+ \subset \mathbb{P}^3$ is a smooth quadric hypersurface, and $\mu \leq n$, $(E^+, \frac{1}{n}D_{E^+}^+)$ not being log canonical is a contradiction [36, proposition 1.2].

Alternatively, since E^+ is a smooth quadric surface, $E^+ \cong \mathbb{P}^1 \times \mathbb{P}^1$, and since a hyperplane section is given by $H_{E^+} \sim F+G$ (where the classes F and G correspond to the two families of line generators on E^+), it follows that $D^+_{E^+} \sim \mu F + \mu G$ is an effective curve of bidegree (μ, μ). This is impossible [40].

Hence, $\mu > n$, and so $\mathrm{mult}_o D > 2n$ by (41) and the fact that E^+ is quadratic. However, the degree of D is

$$\deg D = (D \cdot \pi^* H_\Pi^{l-1}) = 2n$$

where H_Π is the class of a hyperplane in Π, since V_Π is a double cover and $D \sim n\pi^* H_\Pi$. Therefore, we have $\mathrm{mult}_o D > \deg D$, which is not possible. The contradiction proves the theorem.

7 References

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